The average rank of elliptic curves is bounded, over any global field

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Mordell's theorem

Theorem (Mordell 1922)

Let $E/\mathbb{Q}$ be an elliptic curve. Then, $E(\mathbb{Q})$ is finitely generated.

Remark

- This was later generalized to Jacobians of curves over number fields, by Weil.
- Of course, we all know it continues to hold for arbitrary abelian varieties over global fields.

Motivating Question

How does rank $E(K)$ *vary as we vary* E*?*

Average rank over $\mathbb{Q}$

Notation

- For $E\colon y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}$, define $\mathrm{ht}(E) := \max\{\left|4A^3\right|, 27B^2\}$.
- Set $\mathcal{E}_{\mathbb{Q}}(X) := \{E/\mathbb{Q} : \mathrm{ht}(E) < X\}$ and

$$\mathbb{E}[\operatorname{rank} E(\mathbb{Q})] \overset{*}{:=} \limsup_{X\to\infty} \frac{\sum_{E\in\mathcal{E}_{\mathbb{Q}}(X)} \operatorname{rank} E(K)}{\#\mathcal{E}_{\mathbb{Q}}(X)}.$$

Theorem (Bhargava–Shankar 2013)

$$\mathbb{E}[\operatorname{rank} E(\mathbb{Q})] \leq 0.885.$$

In fact, $\mathbb{E}[\#\operatorname{Sel}_n(E/\mathbb{Q})] = \sum_{d|n} d$ *if* $n = 1, 2, 3, 4, 5.$

Average rank more generally

Theorem (Shankar 2013)

Let K be a number field. Then,

$$\mathbb{E}[\#\operatorname{Sel}_2(E/K)] \leq 3 \text{ and } \mathbb{E}[\operatorname{rank} E(K)] \leq \frac{3}{2}.$$

Theorem (Hồ–Lê Hùng–Ngô 2014, A. 2023)

Let K be a global function field with $\mathbb{F}_q$ as its field of constants. Then,

$$\mathbb{E}[\#\operatorname{Sel}_2(E/K)] \leq 3 + O_{g(K)}\left(\frac{1}{q}\right) \text{ and } \mathbb{E}[\operatorname{rank} E(K)] \leq \frac{3}{2} + O_{g(K)}\left(\frac{1}{q}\right)$$

as $q \to \infty$.

Corollary

For any global field K, $\mathbb{E}[\operatorname{rank} E(K)] < \infty$.