Arithmetic Strength of Curves

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Def: the *strength* of a homog. poly. $f \in k[x_0, \ldots, x_n]$ is the least *r* such that we can write

$$f = g_1 h_1 + g_2 h_2 + \cdots + g_r h_r$$

with $\deg(g_i), \deg(h_i) \geq 1$.

The *collective strength* of a collection of poly.s $\{f_1, \ldots, f_k\}$ is the minimum strength of a homog. linear combo. of the f_i.

E.g., over $\mathbb{C}$

$$\mathrm{str}(x^2 + y^2) = 1 \qquad \mathrm{str}(x^2 + y^2 + z^2) = 2$$

$$\mathrm{str}(x^2 + y^2 + z^2, 2xy + z^2) = 1$$

We always have $\mathrm{str}(f) \leq n + 1$

Observation: (Ellenberg) if the hypersurface $V(f)$ has a rational point, then $\mathrm{str}(f) \leq n$.

$\Rightarrow$ "*f* has bounded strength" generalizes "$V(f)$ has a rational point".

Questions:

1. Fix *n*, *d*, and *r*. What is the density of deg. *d* poly.s in *n+1* var.s with $\mathrm{str} \leq r$ over $\mathbb{Q},\ \mathbb{R},\ \mathbb{F}_p,\ \mathbb{Q}_p,\ \ldots$?
2. Do those poly.s satisfy a Hasse principle?

Results so far: for 2 quadrics in 3 variables, collective strength is controlled by the base locus over $\overline{k}.$ Related to Bhargava's work on quartic discriminants.

- Over any field, $\mathrm{str}(f_1, f_2) \leq 2.$
- Over $\mathbb{R}$, 50% strength 2, 50% strength 1.
- Over $\mathbb{Q}$, 100% strength 2.
- Over $\mathbb{F}_q$, about 7/12 strength 2. Exactly:

$$\frac{7 \cdot \#\,\mathrm{PGL}_3(\mathbb{F}_q)}{12 \cdot \#\,\mathrm{Gr}(2,6)_q}$$

- Over $\mathbb{Q}_p$, similar proportion for large *p*, different for $p = 2, 3.$

Further directions:

- pairs of quadrics in higher dimension
- pairs of cubic curves
- analogs over function fields

Question: is this related to $\mathbb{A}^1$-homotopy theory?

Thank you