The tale of the Fibration Method

① The Quest of Rational Points

Challenge

If I give you a variety $X/\mathbb{Q}$, tell me if $X(\mathbb{Q}) \neq \emptyset$!

Hasse's suggestion:

$$X(\mathbb{Q}) \neq \emptyset \iff X(\mathbb{R}) \neq \emptyset \ \& \ X(\mathbb{Q}_p) \neq \emptyset \ \forall \text{prime } p$$

↳ TRUE for CONICS (Hasse - Minkowski 1921)

↳ FALSE in general

$\dim X = 1 \rightsquigarrow 2y^2 = x^4 - 17 \neq 0$ (Lind - Reichardt '40)

$\dim X = 2 \rightsquigarrow y^2 + z^2 = (3 - x^2)(x^2 - 2)$ (Iskovskikh '71)

2 Manin's suggestion

Brauer-Manin pairing

. $X/\mathbb{Q}$, $X(\mathbb{A}_\mathbb{Q}) := \prod\limits_{v \in \Omega_\mathbb{Q}} X(\mathbb{Q}_v)$, $\mathrm{Br}(X) := H^2_{et}(X, \mathbb{G}_m)$

$\Omega_\mathbb{Q} \rightsquigarrow$ {places of $\mathbb{Q}$}

$$X(\mathbb{A}_\mathbb{Q}) \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$$
$$((x_v), \alpha) \longmapsto \sum_{v \in \Omega_\mathbb{Q}} \mathrm{inv}_v \, x_v^* \alpha$$

Brauer-Manin set : $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}(X)} \supset X(\mathbb{Q})$

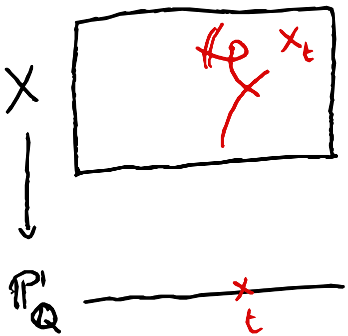
$\hookrightarrow$ class field theory

Conjecture: if X is proper, smooth and rationally connected then:

$$X(\mathbb{Q}) \neq \emptyset \iff X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}(X)} \neq \emptyset$$ (X verifies GHP)

③ Manin's suggestion in family

Conjecture (fibration method)



. if $f: X \longrightarrow \mathbb{P}^1_{\mathbb{Q}}$ s.t.:

smooth, proper, geom. connected

1) $\forall t \in \mathbb{P}^1(\mathbb{Q})$, X_t verifies (GHP)

2) X_η is smooth & rationally connected

. Then X verifies (GHP)

Theorem (B., in progress)

. let $f: X \to \mathbb{P}^1_{\mathbb{F}_q(t)}$ and B its set of bad fibres s.t.:

1) $\forall t \in \mathbb{P}^1(\mathbb{F}_q(t))$, X_t verifies (GHP)

2) X_η smooth & separably rationally connected

. Then $X(\mathbb{A}_{\mathbb{Q}})^{Br(X)} \neq \emptyset \Rightarrow \forall n \gg_{\deg(B)} 1$, $X(\mathbb{F}_{q^n}(t)) \neq \emptyset$

Thank you!