The motive of the g-fold product of a CM elliptic curve

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The Mordell conjecture 100 years later
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Motives

- Let k be a field and X/k a smooth projective variety.
- There is a functor sending X to its Chow motive $\mathfrak{h}(X)$.
- The functors sending X to its ℓ-adic cohomology $H^\bullet(X_{\overline{k}}, \mathbb{Q}_\ell)$, and to its Chow group of algebraic cycles (with $\mathbb{Q}$-coefficients) $\mathrm{CH}^\bullet(X)$, both factor through the motive.
- The category of motives has Tate twists, direct sums and a tensor product, with $\mathfrak{h}(X \times_k Y) \cong \mathfrak{h}(X) \otimes_{\mathbb{Q}} \mathfrak{h}(Y)$.

The motive of a CM elliptic curve

- The motive of an abelian variety A/k, with $\dim A = g$, has a decomposition

$$\mathfrak{h}(A) \cong \mathfrak{h}^0(A) \oplus \mathfrak{h}^1(A) \oplus ... \oplus \mathfrak{h}^{2g}(A)$$

  which induces the grading on ℓ-adic cohomology (Deninger-Murre, 1991).
- The decomposition satisfies a Kunneth-type formula with respect to products of abelian varieties.
- Let E be an elliptic curve. The motive of E^g decomposes into Tate twists of motives of the form $\otimes^r_{\mathbb{Q}}\mathfrak{h}^1(E)$.
- When E has CM by a quadratic imaginary field K, the decomposition refines into Tate twists of motives of the form $\otimes^r_K\mathfrak{h}^1(E)$.

Chow groups of $\otimes^g_K\mathfrak{h}^1(E)$

- Moonen asked whether $\mathrm{CH}^i(\otimes^g_K\mathfrak{h}^1(E)) = 0$ when $i \neq g$ (Moonen's original question restricted to char$(k) > 0$, but we let k be arbitrary here).
- We show that

$$\mathrm{CH}^2(\otimes^3_K\mathfrak{h}^1(E))/\sim_{alg} \cong \mathrm{Griff}(E^3) := \mathrm{CH}^2(E^3)_0/\sim_{alg} .$$

- Bloch in 1984 gave an example of $E/\mathbb{Q}(\zeta_8)$ with $\mathrm{Griff}(E^3) \neq 0$, so the answer is no over number fields.

Bloch-Beilinson conjecture

- Bloch was motivated by

Conjecture

When k is a number field,

$$\dim_{\mathbb{Q}} \mathrm{CH}^i(X)_0 = \mathrm{ord}_{s=i} L(H^{2i-1}(X_{\overline{k}}, \mathbb{Q}_\ell), s).$$

- There is also a refined form of the conjecture.
- We pose the same conjecture when k is a global function field.
- For certain X/k, the L-function appearing above can then be computed explicitly.

Motives over function fields

- When E/k has CM, we show that

$$\dim_{\mathbb{Q}} \mathrm{CH}^i(E^g)_0 \geq \mathrm{ord}_{s=i} L(H^{2i-1}(E^g_{\overline{k}}, \mathbb{Q}_\ell), s).$$

  We also show that part of the the refined conjecture holds for E^2, and the rest of the conjecture for E^2 follows from other standard conjectures about motives.
- The conjecture predicts that

$$\dim_{\mathbb{Q}} \mathrm{CH}^i(\otimes_K^g \mathfrak{h}^1(E)) = 0$$

  for all i and all $g \geq 2$, which would affirm Moonen's question in this case.