Isogeny relations in products of families of elliptic curves

Luca Ferrigno

Università degli studi Roma Tre

The Mordell conjecture 100 years later
July 9, 2024

General setting

Let $\mathcal{E}_\lambda \to Y(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the Legendre family of elliptic curves, i.e. the family with generic fiber

$$E_\lambda : Y^2 Z = X(X - Z)(X - \lambda Z)$$

and let $\mathcal{E}_\lambda^n \to Y(2)$ be its n-fold fibered power.
Consider a curve $C \subseteq \mathcal{E}_\lambda^n \times \mathcal{E}_\mu^n \to Y(2) \times Y(2)$, then each point $\mathbf{c} \in C(\mathbb{C})$ corresponds to m points $P_1(\mathbf{c}), \ldots, P_m(\mathbf{c})$ on $E_{\lambda(\mathbf{c})}$ and n points $Q_1(\mathbf{c}), \ldots, Q_n(\mathbf{c})$ on $E_{\mu(\mathbf{c})}$.
Suppose also that the points P_i and Q_j are linearly independent on C, that is, there is non non-trivial relation of the form

$$a_1 P_1 + \ldots + a_m P_m = O \quad \text{or} \quad b_1 Q_1 + \ldots + b_n Q_n = O$$

with $a_i \in \mathrm{End}\left(\mathcal{E}_\lambda|_C\right)$ and $b_j \in \mathrm{End}\left(\mathcal{E}_\mu|_C\right)$, that holds generically on C.

Some known results

Let $C \subseteq \mathcal{E}_\lambda^n \times \mathcal{E}_\mu^n \to Y(2) \times Y(2)$ be an irreducible curve, defined over $\overline{\mathbb{Q}}$, with λ and μ not both constant. Suppose moreover that, on C, E_λ and E_μ are not generically isogenous and that the points P_i and Q_j are generically independent.

Theorem (Masser-Zannier, 2014)

Let $C \subseteq \mathcal{E}_\lambda \times \mathcal{E}_\mu \to Y(2) \times Y(2)$ as above. Then there are at most finitely many $\mathbf{c} \in C(\mathbb{C})$ such that $P(\mathbf{c}) \in E_{\lambda(\mathbf{c})}$ and $Q(\mathbf{c}) \in E_{\mu(\mathbf{c})}$ are both torsion.

Theorem (Barroero-Capuano, 2017)

Let $C \subseteq \mathcal{E}_\lambda^n \times \mathcal{E}_\mu^n \to Y(2) \times Y(2)$ as above. Then there are at most finitely many points $\mathbf{c} \in C(\mathbb{C})$ such that there exist $a_1, \ldots, a_m \in \operatorname{End}(\mathcal{E}_\lambda|_C)$ and $b_1, \ldots, b_n \in \operatorname{End}(\mathcal{E}_\mu|_C)$ for which

$$a_1 P_1(\mathbf{c}) + \ldots + a_m P_m(\mathbf{c}) = O \quad \text{and} \quad b_1 Q_1(\mathbf{c}) + \ldots + b_n Q_n(\mathbf{c}) = O.$$

Isogeny relations in products of families

Theorem (F., 2024)

Let $C \subseteq \mathcal{E}^m \times \mathcal{E}^n \to Y(2) \times Y(2)$ be a asymmetric irreducible curve defined over $\overline{\mathbb{Q}}$ with λ and μ not both constant. Suppose moreover that E_λ and E_μ are not generically isogenous and that no relation of the form

$$a_1 P_1 + \ldots + a_m P_m = O \text{ or } b_1 Q_1 + \ldots + b_n Q_n = O$$

holds identically on C. Then, there are at most finitely many $\mathbf{c} \in C(\mathbb{C})$ such that there exists an isogeny $\phi : E_{\mu(\mathbf{c})} \to E_{\lambda(\mathbf{c})}$ and there exists $(a_1, \ldots, a_m, b_1, \ldots, b_n) \in \mathrm{End}(E_{\lambda(\mathbf{c})})^{m+n}$ not all zero with

$$a_1 P_1(\mathbf{c}) + \ldots + a_m P_m(\mathbf{c}) + b_1 \phi(Q_1(\mathbf{c})) + \ldots + b_n \phi(Q_n(\mathbf{c})) = O.$$