How do generic properties spread?

Families of isogenous elliptic curves ordered by height

Jerry Yu Fu

California Institute of Technology

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Table of Contents

Jerry Yu Fu

The General Question

- Given a family $\mathcal{X} \to \mathcal{S}$ of algebraic varieties over a field k, with $\mathcal{S}$ an irreducible scheme. Let $\mathcal{X}_\eta$ be its generic fiber.
- **Question:**
  - What type of properties of $\mathcal{X}_\eta$ extend to other fibers? [e.g., smoothness, (geometrically) simple, cohomology, Picard rank...]
  - How/In which way do these properties extend? Can we get a quantitative estimation for the 'exotic' points?
- **Hilbert irreducibility theorem:** for a number field k, a dominant map $X \to \mathbb{P}^n$ defined over k which is generically of degree d, the fiber over 'most' k-rational points $t \in \mathbb{P}^n(k)$ is a finite set of Galois-conjugate points where G acts freely transitively.
- **Quantitative estimates for size of the complement(S. Cohen):**

$$|M_k(B)| = O\left(B^{(n-1/2)d}(\log B)^\gamma\right)$$

with $\gamma < 1$, $[k:\mathbb{Q}] = d$.

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Table of Contents

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- ⋆ Define ι to be the map:

$$\iota : X(1) \times X(1) \supset C \to \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3.$$

such that ι is the composition of j-invariant map and the Segre embedding.

- ⋆ Let $H(P_t)$ be the projective height of $\iota(P_t) \in \mathbb{P}^3$.
- ⋆ Let $S(B)$ be the set of specializations $t \in C(K)$ where there is an $\overline{\mathbb{Q}}$-isogeny between E_t and E'_t with height at most B.

Theorem (Fu, 2023)

Let K be a number field of degree d_K. let C be a rational curve over K isomorphic to $\mathbb{P}^1$ which parametrizes a one-dimensional family of pairs of elliptic curves (E, E'). Let (E_t, E'_t) be the generic fiber of this family over $K(t)$, and suppose that there exists no $\overline{K(t)}$-isogeny between E_t and E'_t. Let $d = \deg \iota^* \mathcal{O}_{\mathbb{P}^3}(1)$ be the degree of the parameter family C defined with respect to ι. We have

$$|S(B)| \lesssim_K d^4 (\log B)^6.$$

Table of Contents

Jerry Yu Fu

Relate to the uniform boundedness conjecture

- ⋆ Let $Z(C;B)(K) = \{x \in C \cap \bigcup_n X_0(n)(K) \mid H(x) \leq B\}$ Our main theorem can be reformulated as: $|Z(C;B)(K)| \lesssim_K d^4(\log B)^6$
- ⋆ **Uniform boundedness(Merel,1994):** Suppose $[K:\mathbb{Q}] = d$, we have $|E(K)_{\text{tors}}| \leq B(d)$.
- ⋆ (Parent, 1999) E/K, $[K:\mathbb{Q}] = d$, $E[p^n](K) \neq O$. Then

$$p^n \leq \begin{cases} 129\left(3^d-1\right)(3d)^6 & \text{if } p=2, \\ 65\left(5^d-1\right)(2d)^6 & \text{if } p=3, \\ 65\left(3^d-1\right)(2d)^6 & \text{if } p \geq 5. \end{cases}$$

- ⋆ $d = 1$, Mazur. $B(2) = 24$ vs Parent's bound $\Rightarrow 6.3 \times 10^{39}$ Not sharp!
- ⋆ Our theorem describes the distribution of K-rational points on $\bigcup_n X_0(n)$ cut out by C, in terms of height, over arbitrary number field K.

Question

- ⋆ Can we remove the dependence on C? Or at least the degree of C?
- ⋆ Suppose C is a curve defined over $\overline{\mathbb{Q}}$. Is $Z(C;B)(\overline{Q})$ finite? If this is the case, can we get an upper bound of $Z(C;B)(\overline{Q})$ in terms of B?