# Towards a classification of $p^2$-discriminant ideal twins over number fields

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# Background and Motivation

**Definition**

Let $K$ be a number field and let $E$ and $E'$ be elliptic curves defined over $K$ that are not $K$-isomorphic. We say that $E$ and $E'$ are **discriminant ideal twins** if they have the same discriminant ideal and the same conductor.

If, in addition, for each prime $\mathfrak{p}$ there exist $\mathfrak{p}$-minimal models for $E$ and $E'$ defined over $\mathcal{O}_K$ such that $\Delta_{\mathfrak{p}} = \Delta'_{\mathfrak{p}}$, then we say $E$ and $E'$ are **discriminant twins**.

Two isogenous elliptic curves have the same conductor.

**Question:** When can two isogenous non-isomorphic elliptic curves defined over a number field have the same minimal discriminant ideal?

# Previous Work

**Question:** When can two isogenous non-isomorphic elliptic curves defined over a number field have the same minimal discriminant ideal?

- (Deines, 2018) Over $\mathbb{Q}$, there are only finitely many semistable (i.e., multiplicative at each prime) isogenous discriminant twins.
- (Barrios, Brucal-Hallare, Deines, Harris, Roy, 2024) The authors provide a classification of all $p$-isogenous discriminant ideal twins over number fields where $p \in \{2, 3, 5, 7, 13\}$, i.e., where $X_0(p)$ has genus 0.
- If $E_1$ and $E_2$ are $n$-isogenous elliptic curves over $K$, then there are $t \in K$ and $d \in \mathcal{O}_K$ such that $E_i$ is $K$-isomorphic to $C_{n,i}(t,d)\colon y^2 = x^3 + d^2 A_{n,i}(t)x + d^3 B_{n,i}(t)$.

# Main Theorems

## Theorem (Deines, H., Iorga, Namoijam, Roy, Watson, 2024)

*Let $E_1$ and $E_3$ be $p^2$-isogenous elliptic curves over a number field $K$ such that their $j$-invariants are not equal. Suppose further that $E_i \cong C_{p^2,i}(t,d)$, where $t \in \mathcal{O}_K$ and $d \in \mathcal{O}_K/(\mathcal{O}_K)^2$. Then $E_1$ and $E_3$ are discriminant ideal twins if and only if for each prime $\mathfrak{p}$ of $\mathcal{O}_K$,*

- $\underline{p = 3}$: *we have $\nu_{\mathfrak{p}}(t-3) = 3k_{\mathfrak{p}}$ for $0 \le k_{\mathfrak{p}} \le \nu_{\mathfrak{p}}(3)$.*

  *The two curves are discriminant twins if and only if $t$ satisfies the above and $(t-3)^8 \in \mathcal{O}_K^{12}$.*
- $\underline{p = 5}$: *we have $\nu_{\mathfrak{p}}(t-1) = k_{\mathfrak{p}}$ for $0 \le k_{\mathfrak{p}} \le \nu_{\mathfrak{p}}(5)$.*

  *The two curves are discriminant twins if $t$ satisfies the above.*

## Corollary (Deines, H., Iorga, Namoijam, Roy, Watson, 2024)

*Up to twists, there are finitely many $p^2$-isogenous discriminant ideal twins over $\mathbb{Q}$, for odd $p$ for which $X_0(p^2)$ has genus 0.*