Heegner points on $y^2 = x^3 + p$

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The Mordell Conjecture 100 Years Later
Massachusetts Institute of Technology
July 8, 2024

The Problem

Consider the elliptic curve E_D with equation $y^2 = x^3 + D$. When does this curve have nontorsion rational points?

- *Sylvester's problem* deals with the case $D = -27k^2$. Here, E_D is isomorphic to the curve $x^3 + y^3 = 2k$.
- Much work has been done to show that there nontrivial rational points on $x^3 + y^3 = k$ for many integers k. (Satgé (1987), Elkies (1994), and others)
- All of these are *cubic twists* of the curve $y^2 = x^3 + 1$. What about twists of degree 6?

Heegner Points

Let $E_D \colon y^2 = x^3 + D$ and $\widetilde{E}_D \colon x^3 + y^3 = D$.
Our main result is the following.

THEOREM. *Let p be a prime congruent to $5 \pmod 9$, and let $\epsilon = (-1)^{(p-1)/2}$. If $K := \mathbb{Q}(\sqrt[3]{p})$ has odd class number, $E_{\epsilon p}$ has rank 1 and $E_{-\epsilon p}$ has rank 0.*

What is the Heegner point construction here? We construct a modular parametrization (X, Y) of level 6 for E_1 from division values of the Weierstrass $\wp$-function. Look at

$$E_1 \to E_{-27} \to \boxed{\widetilde{E}_2 \to C_{p^2} \to \widetilde{E}_{2p^2}} \to \boxed{E_{-27p^4} \to E_{p^4} \to E_{\epsilon p}}$$

where C_{p^2} is the curve with equation $p^2x^3 + 2y^3 = 1$.

Nontriviality

Why is this point nontrivial?

The challenge – *how do we incorporate the condition that h_K is odd?* (This is necessary because E_p can have large rank, but Cassels showed that $\operatorname{rk} E_D + \operatorname{rk} E_{-D} \leq 1$ when $h_{\mathbb{Q}(\sqrt[3]{D})}$ is odd and D is cubefree.)

Our next theorem answers this question.

Theorem. *If u is the unique fundamental unit of $\mathcal{O}_K$ greater than 1, $N_{R_{6p}/K}(X(p\omega)+1) = 3^{p+1}u^{3h_K}$.*

When h_K is odd, the right hand side is not a square in R_{6p}, so our Heegner point is nontrivial. Moreover, this point is nontrivial in $E_{\epsilon p}(\mathbb{Q})/2E_{\epsilon p}(\mathbb{Q})$, so it is an odd multiple of the generator.

Thank you!