THE DISTRIBUTION OF SELMER GROUPS IN QUADRATIC TWIST FAMILIES OVER FUNCTION FIELDS

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1. Background on Elliptic curves

Let $K = \mathbb{F}_q(t)$ of characteristic > 3. We can write elliptic curves E as $E : y^2 = x^3 + Ax + B$, for $A, B \in \mathbb{F}_q[t])$.

Theorem 1.1 (Mordell-Weil). *The group of K-rational points $E(K) \simeq \mathbb{Z}^r \oplus T$ where T is a finite abelian group.*

The number r is the **rank** of E.

Question 1.2 (Motivating Question). How often does E have finitely many solutions? More generally, what is the average value of r?

To make sense of average size, we'll need to work with a specific family of elliptic curves.

Definition 1.3. Given a fixed elliptic curve $E : y^2 = x^3 + Ax + B$, we can work with the *quadratic twist family*

$$\{E_f := f(t)y^2 = x^3 + A(t)xz^2 + B(t)z^3\}$$

for $f(t) \in \mathbb{F}_q[t]$ squarefree and prime to the discriminant of E. We define $\mathrm{QTwist}_n(\mathbb{F}_q)$ to be the set of polynomials as above with degree n. (For convenience, we will restrict to f having even degree.)

Conjecture 1.4 (Bhargava–Shankar [BS13, Conjecture 4] and Poonen–Rains [PR12, Conjecture 1.4(b)]). When elliptic curves are ordered by height,

$$\lim_{n\to\infty} \mathbb{E}_{f\in \mathrm{QTwist}_n(\mathbb{F}_q)}(\#\mathrm{Sel}_\ell(E_f)) = \ell + 1.$$

Theorem 1.5. *Suppose E has a fiber of multiplicative reduction.*

$$\lim_{j\to\infty} \limsup_{n\to\infty} \mathrm{Prob}_{f\in \mathrm{QTwist}_n(\mathbb{F}_{q^j})}(\mathrm{rk}(E_f) = r) = \begin{cases} 1/2 & \text{if } r \in \{0,1\} \\ 0 & \text{if } r \geq 2, \end{cases}$$

and similarly for liminf in place of limsup. Also, for ℓ avoiding an explicit finite set of primes,

$$\lim_{j\to\infty} \lim_{n\to\infty} \mathbb{E}_{f\in \mathrm{QTwist}_n(\mathbb{F}_{q^j})}(\#\mathrm{Sel}_\ell(E_f)) = \ell + 1.$$

References

[BS13] Manjul Bhargava and Arul Shankar. The average number of elements in the 4-selmer groups of elliptic curves is 7. *arXiv preprint arXiv:1312.7333v1*, 2013.

[PR12] Bjorn Poonen and Eric Rains. Random maximal isotropic subspaces and Selmer groups. *J. Amer. Math. Soc.*, 25(1):245–269, 2012.