Affine Chabauty

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The Mordell conjecture 100 years later, Cambridge, MA

Results

- $Y = X \setminus D$ affine curve, where
- X smooth projective curve over $\mathbb{Q}$ of genus g.
- $\emptyset \neq D \subset X$ divisor with $n = \#D(\overline{\mathbb{Q}}) = \#D(\mathbb{Q})$.
- J the Jacobian of X, $r = \operatorname{rk} J(\mathbb{Q})$.
- p a prime of good reduction for Y.

Theorem (L.–Lüdtke–Müller, 2023)

If $r < g + n - 1$, then $(Y(\mathbb{Z}) \subset)$ $Y(\mathbb{Z}_p)_1$ is finite.

If $\frac{1}{2}r(r+3) < \frac{1}{2}g(g+3) + n - 1$, then

$$\#Y(\mathbb{Z}_p)_1 \leq \kappa_p \cdot \prod_{\ell} n_\ell \cdot \#Y(\mathbb{F}_p) \cdot (4g + 2n - 2)^2 (g+1)\,.$$

Method: Abelian Chabauty–Kim

Let $U = \pi_1(Y_{\overline{\mathbb{Q}}})^{ab}_{\mathbb{Q}_p}$. Then

$$\begin{array}{ccc} Y(\mathbb{Z}) & \hookrightarrow & Y(\mathbb{Z}_p) \\ \downarrow & & \downarrow j_p \\ Sel_U(\mathbb{Q}_p) & \xrightarrow[\text{loc}_p]{} & H^1_f(G_p, U)(\mathbb{Q}_p) \end{array}$$

Define $Y(\mathbb{Z}_p)_1 := j_p^{-1}(im(\text{loc}_p)) \subset Y(\mathbb{Z}_p)$. Use

$$1 \longrightarrow \mathbb{Q}_p(1)^{n-1} \longrightarrow U \longrightarrow V_pJ \longrightarrow 1$$

to calculate the local and global Selmer dimensions:

	global	local
V_pJ	r_p	g
$\mathbb{Q}_p(1)^{n-1}$	0	$n-1$

What's next?

Let J_D be the generalised Jacobian of Y.

$$\begin{array}{ccccc} Y(\mathbb{Z}) & \hookrightarrow & Y(\mathbb{Z}_p) & & \\ \downarrow & & \downarrow j_p & \searrow & \\ J_D(\mathbb{Z}) & \hookrightarrow & J_D(\mathbb{Z}_p) & \xrightarrow{\log} & H^0(X_{\mathbb{Q}_p}, \Omega_{3,D})^\vee. \end{array}$$

Hope

If $r < g + n - 1$ and p is large enough, then

$$\#Y(\mathbb{Z}_p)_1 \leq \#Y(\mathbb{F}_p) + (2g - 2 + n).$$

Thank you for your attention!