Cuspidal Subgroups of Modular Jacobians

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8/7/2024

Cuspidal Subgroups

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup and Y the algebraic curve associated to the quotient $\Gamma\backslash\mathbb{H}$. This curve is compactified by adding finitely many points, known as *cusps*

$$X = Y \cup \{c_1, \ldots, c_n\}.$$

The *cuspidal subgroup* of X is the subgroup of the Jacobian J of X generated by the cusps

$$C = \langle [c_i - c_j] \ : \ 1 \leq i, j \leq n \rangle.$$

Theorem (Manin '72 - Drinfeld '73)

If Γ is a congruence subgroup, C is a finite.

In particular, $C \subset J(K)_{\text{tors}}$, where K is the field of definition of the cusps.

When Γ is a congruence subgroup, what is C ?

Some Results

Let $p \geq 5$ prime and $\Gamma = \Gamma_0(p)$. The curve $X = X_0(p)$ has two cusps c_1, c_2, both defined over $\mathbb{Q}$.

Theorem (Ogg '73)

$C = C_0(p) \cong (\mathbb{Z}/N_p\mathbb{Z})$, *where* N_p *is the numerator of* $(p-1)/12$.

Theorem (Mazur '77)

$C_0(p) = J_0(\mathbb{Q})_{tors}$.

- rational cuspidal divisor class group of $X_0(N)$ (Yoo '22)
- cuspidal divisor class number, rational divisor class number of $X_1(p)$(Yu '80, Takagi '92)
- $J_1(p)(\mathbb{Q})_{\text{tors}}$ is cuspidal up to $2-$torsion (Ohta '13)

A Covering of $X_0(p)$

Let p be a prime, $p \equiv 1 \bmod 4$ and let $H \subset (\mathbb{Z}/p\mathbb{Z})^*$ be the subgroup of squares. We write $X_H(p)$ for the complete curve corresponding to the congruence subgroup

$$\Gamma_H(p) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \;:\; a \bmod p \in H, c \equiv 0 \bmod p\}.$$

The curve $X_H(p)$ has 4 cusps c_1, c_2 (over $\mathbb{Q}$) and c_3, c_4 (over $\mathbb{Q}(\sqrt{p})$).

Theorem (L.,Yoo '24$^+$)

- $C_H^\infty(p) = \langle [c_1 - c_2] \rangle \cong \mathbb{Z}/m_p\mathbb{Z}$
- $C_H^\mathbb{Q}(p) = \langle [c_1 - c_2], [c_1 + c_2 - c_3 - c_4] \rangle \cong \mathbb{Z}/m_p\mathbb{Z} \times \mathbb{Z}/n_p\mathbb{Z}$
- $C_H^\mathbb{Q}(p) = C_H(p)(\mathbb{Q})$
- $C_H(p) = \langle [c_1 - c_2], [c_1 - c_3], [c_1 - c_4] \rangle \cong (\mathbb{Z}/m_p\mathbb{Z})^2 \times \mathbb{Z}/n_p\mathbb{Z}$

where $m_p = \frac{p}{2} \sum_{i=0}^{\frac{p-3}{2}} (-1)^i B_2\left(\frac{\alpha^i}{p}\right)$, $B_2(x) = \{x\}^2 - \{x\} + \frac{1}{6}$, α *is an integer of order* $p-1$ *modulo* p *and* n_p *is the numerator of* $p - 1/24$.