Arakelov canonical divisor and Bogomolov conjecture for modular curves

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The Mordell conjecture 100 years later
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Setting

Let X be a smooth, geometrically integral, projective curve defined over a field F, with model of X we mean an integral, projective, flat, 2-dimensional scheme over $\mathrm{Spec}\, \mathcal{O}_F$ with a fixed isomorphism between its generic fiber and X.

We are interested in the classical modular curves $X_0(N)$ for genus $g \geq 2$.

Arakelov divisors allow us to define an intersection pairing $\langle \cdot, \cdot \rangle$ on an arithmetic surface that descends to the quotient with respect to principal divisors.

There is a canonical Arakelov divisor ω satisfying properties similar to the ones satisfied by the classical canonical divisor. We are interested in its self-intersection $\omega^2 := \langle \omega, \omega \rangle$.

To prove Bogomolov conjecture, Zhang in 1993 introduced a modified Arakelov divisor ω_a, the admissible ω, showing that

$$\omega^2 \geq \omega_a^2 \geq 0.$$

Asymptotic behaviour

Theorem (Dolce, M., 2024)

For the minimal regular model over $\operatorname{Spec}\mathbb{Z}$ *of* $X_0(N)$ *we have:*

$$\omega^2 \sim 3g \log N, \qquad \text{for } N \to +\infty \text{ and } (N,6) = 1.$$

We have $\omega_a^2 = \omega^2 - r$, with $r \geq 0$.

Theorem (Michel, Ullmo, 1998)

For the semistable minimal regular model over $\operatorname{Spec}\mathbb{Z}$ *of* $X_0(N)$*, with* N *square-free, we have:*

$$r \sim g/3, \qquad \text{for } N \to +\infty \text{ and } (N,6) = 1.$$

Theorem (Banerjee, Chaudhuri, 2021)

For the stable model over $\operatorname{Spec}\mathcal{O}_L$ *of* $X_0(p^2)$*, with* p *prime, we have:*

$$\begin{aligned} \omega^2 &\sim 2g \log p^2, \\ r &\sim 0, \end{aligned} \qquad \text{for } p \to +\infty \text{ and } L = \mathbb{Q}(p^{\frac{2}{p^2-1}}, \zeta_{p+1}).$$

Effective Bogomolov

Theorem

Let X be a geometrically connected smooth curve over a field F with genus $g \geq 2$ and Jacobian J_X, let $\iota_{D_0} : X \to J_X$ be an embedding of the curve in its Jacobian and let h_{NT} be the Néron-Tate height on J_X. Then:

1. *For every $\varepsilon > 0$ and for every degree 1 divisor D_0 the set $\{x \in X(\overline{F}) : h_{\mathrm{NT}}(\iota_{D_0}(x)) < \frac{\langle \omega_a, \omega_a \rangle}{4(g-1)} - \varepsilon\}$ is finite.*
2. *For every $\varepsilon > 0$ and divisor D_0 of degree 1 such that $D_0 - \frac{1}{2g-2}K$ is a torsion point in J_X, with K a canonical divisor of X, the set $\{x \in X(\overline{F}) : h_{\mathrm{NT}}(\iota_{D_0}(x)) < \frac{\langle \omega_a, \omega_a \rangle}{2(g-1)} + \varepsilon\}$ is infinite.*

For $X_0(N)$, with N large enough square-free and coprime with 6, we have:

$$\left\{x \in X_0(N)(\overline{\mathbb{Q}}) : h_{\mathrm{NT}}(\iota_\infty(x)) < \tfrac{2}{3}\log N - \varepsilon\right\} \qquad \text{is finite;}$$

$$\left\{x \in X_0(N)(\overline{\mathbb{Q}}) : h_{\mathrm{NT}}(\iota_\infty(x)) < \tfrac{4}{3}\log N + \varepsilon\right\} \qquad \text{is infinite.}$$

For $X_0(p^2)$, with p prime large enough, we have:

$$\left\{x \in X_0(p^2)(\overline{\mathbb{Q}}) : h_{\mathrm{NT}}(\iota_\infty(x)) < \tfrac{1}{2}\log p^2 - \varepsilon\right\} \qquad \text{is finite;}$$

$$\left\{x \in X_0(p^2)(\overline{\mathbb{Q}}) : h_{\mathrm{NT}}(\iota_\infty(x)) < \log p^2 + \varepsilon\right\} \qquad \text{is infinite.}$$