Generalized Campana points and adelic approximation

Boaz Moerman

Utrecht University

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M-points

Study special sets of rational points on a proper variety X over a global field (with an integral model).

Given a rational point and a prime divisor D, have an intersection multiplicity $n_v(D, P)$ at every finite place v.

Let M be a collection of prime divisors $D_1, \ldots, D_n$ together with a set $\mathfrak{M} \subset \mathbb{N}^n$ of allowed multiplicities.

M-points over the S-integers $\mathcal{O}_S$ are the rational points such that $(n_v(D_1, P), \ldots, n_v(D, P)) \in \mathfrak{M}$ for every place v not in S.

Similarly also get v-adic M-points.

If $X = \mathbb{P}^{n-1}$ and $D_i = \{X_i = 0\}$, $1 \leq i \leq n$, then M-points over $\mathbb{Z}$ are $(x_1 : \cdots : x_n)$ such that the valuations lie in $\mathfrak{M}$. For example, fix integers $m_1, \ldots, m_n$, then the M-points are

1. all x_i squarefree if $\mathfrak{M} = \{0, 1\}^n$,
2. x_i m_i-full if $\mathfrak{M} = \{(w_1, \ldots, w_n) \mid w_i \geq m_i \text{ or } = 0\}$ (*Campana points*),
3. $|x_i|$ is an m_i-th power if $\mathfrak{M} = m_1\mathbb{N} \times \ldots m_n\mathbb{N}$. (*Darmon points*)

M-approximation

Let $T \subset S$ be a finite set of places. Consider analogue weak/strong approximation: (X, M) satisfies M-approximation if

$$(X, M)(\mathcal{O}_S) \hookrightarrow \prod_{v \notin S} (X, M)(\mathcal{O}_v) \times \prod_{v \in S} X(K_v)$$

is dense. Similarly, it satisfies M-approximation off T if dense after removing places in T from the product. If X is a split toric variety over a global field and the D_i are the torus invariant prime divisors, then naturally obtain monoids N_M^+ and N_M in the lattice of cocharacters.

Theorem (M. 2024)

Let $T \neq \emptyset$.

- (X, M) satisfies M-approximation iff $N_M^+ = N$,
- (X, M) satisfies M-approximation off T iff $N_M = N$.

N/N_M is a "fundamental group" of (X, M).

Consequences

- The squarefree points ($\mathfrak{M} = \{0,1\}^n$) and Campana points satisfy M-approximation on any split toric variety.
- The Darmon points on $\mathbb{P}^n$ satisfy M-approximation if and only if $\gcd(m_i, m_j) = 1$ for all $i \neq j$.
- Over a number field: a split toric variety satisfies strong approximation off T if and only if it is simply connected. It satisfies strong approximation iff its global sections are constant.