(Toward) An Algorithm to (Explicitly) Produce a Regular Model of a Hyperelliptic Curve in (Bad) Mixed Characteristic (0, 2): A Criterion to Verify Regularity of the Normalization of a Candidate Model

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a "lightning talk" @ The Mordell Conjecture 100 Years Later

Monday, July 8 2024

Regular Model = Normalization of Obus-Srinivasan Model?

Let's find a regular model[1] $\mathcal{Y} \to S := \mathrm{Spec}(\mathscr{O}_K)$ of e.g. the (hyper)elliptic curve $Y \to \eta := \mathrm{Spec}(K)$ described via affine equation[2]:

$$y^2 = f := x^3 + x^2 - 2^7 = (x - \text{“}1\text{”})(x^2 + \epsilon x + \text{“}2^7\text{”}) =: (x - \text{“}1\text{”}) \cdot g.$$

Theorem 7.3 of [LL99] guarantees existence of regular model $\mathcal{X} \to S$ of $X := \mathbb{P}^1 \to \eta$ whose normalization[3] is a regular model $\mathcal{Y} \to S$ of $Y \to \eta$.

A candidate such regular model is an *Obus-Srinivasan* model $\mathcal{X}^* \to S$, inspired by Lemma 2.1 of [Sri15] and subsequent work with Andrew Obus:

- (the reduced scheme associated to) each irreducible component of $\mathrm{div}(f) \hookrightarrow \mathcal{X}^*$ is regular, and
- if ever two irreducible components of $\mathrm{div}(f)$ intersect, then the order of f along (at least) one of them is even.

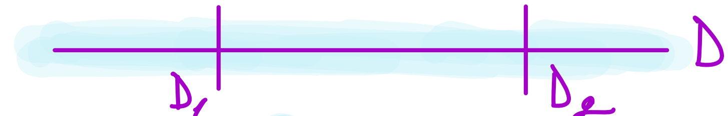
[1] K is a complete, discretely-valued (via v_K) field of characteristic 0 with (local) ring of integers $\mathscr{O}_K$ (with maximal ideal $\mathfrak{m}$) and algebraically closed residue field $k := \mathscr{O}_K/\mathfrak{m}$ of characteristic 2.

[2] “1” denotes the Hensel lift to $\mathscr{O}_K$ of the solution 1 to $f \equiv 0 \pmod 2$, and “2^7” is similar.

[3] in $\mathscr{O}_{\mathcal{X}} \hookrightarrow K(\mathscr{O}_{\mathcal{X}})[y]/(y^2 - f)$

Obus-Srinivasan Model Explicitly via (Inductive) Valuations

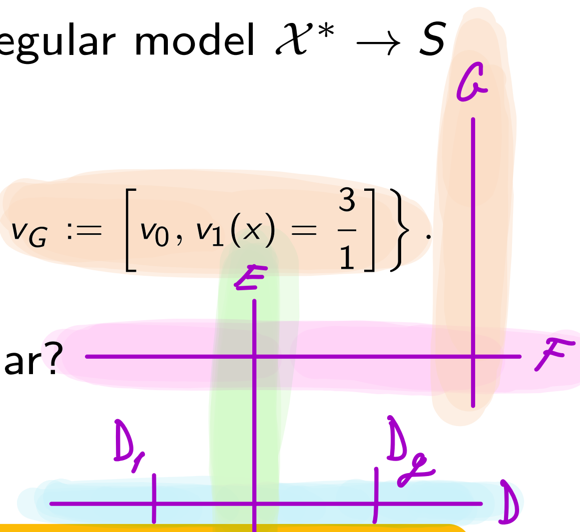
Begin with model $\mathcal{X} \to S$ whose special fiber resembles:



Inspired by Corollary 3.18 of [Rüt14], describe $\mathcal{X} \to S$ explicitly via $\{v_0\}$, where v_0 is the (inductive) *Gauss* valuation[3].

Theorem 4.3 of [OS22] provides explicit description of minimal regular model $\mathcal{X}^* \to S$ along which $\operatorname{div}(f) \hookrightarrow \mathcal{X}^*$ is regular:

$$\left\{ v_D := v_0 = \left[v_0, v_1(x) = \frac{0}{1} \right], v_E := \left[v_0, v_1(x) = \frac{1}{1} \right], v_F := \left[v_0, v_1(x) = \frac{2}{1} \right], v_G := \left[v_0, v_1(x) = \frac{3}{1} \right] \right\}.$$



$\mathcal{X}^* \to S$ is our Obus-Srinivasan model. Is its normalization regular?

The normalization of any point along the generic fiber is regular.

Lemma ("normalization (genuinely) over div(f) is regular" – Myer, to appear in thesis)

Let $\mathcal{X}^* \to S$ be an Obus-Srinivasan model for f. The normalization of a point lying along an irreducible component of $\operatorname{div}(f) \hookrightarrow \mathcal{X}^*$ along which f has odd order is regular.

Otherwise, we might as well assume $p \notin \operatorname{div}(f)$, and by leveraging $k = \bar{k}$, $f_p \in 1 + \mathfrak{m}_p$.

[3] $v_0\left(\sum a_i x^i\right) := \min\{v_K(a_i)\}$

A Criterion for Regularity of the Normalization

Jacobian Criterion reveals only potentially non-regular point along D is p (at $x = 0$).
(Proof of) Lemma 2.3.2 of [CE03] affords us parameters near p:

Lemma ("Local parameters on a regular surface" – 2.3.3 of [CE03])

$R := \hat{\mathscr{O}}^{sh}_{\mathcal{X}^*,p} \cong \hat{\mathscr{O}}^{sh}_{S,s}[[t_1, t_2]]/(t_1^{m_1} t_2^{m_2} \cdot unit - \pi_K)$, *where* $\pi_K^N \cdot Unit = 2$.

Theorem ("Criterion for regularity of normalization" – Myer, to appear in thesis)

The integral closure, S, of $R \hookrightarrow K(R)[y]/(y^2 - f_p)$ (equivalently, the normalization of p in $R \hookrightarrow K(R)[y]/(y^2 - f_p)$) is regular iff either:

- *there exists $q \in S$ such that $trace(q) \notin \mathfrak{m}_p$, or*
- *there exists $q \in S$ such that $norm(q) \in \mathfrak{m}_p - \mathfrak{m}_p^2$.*

Corollary ("Corollarieron" – Myer, to appear in thesis)

$\psi_p \in \mathfrak{m}_p - \mathfrak{m}_p^2$ *for "optimal"* $f_p = 1 + t_1^{M_1} t_2^{M_2} \psi_p \implies$ *the normalization of p is regular.*

$$f = x^3 + x^2 - 2^7 = x^2\left(x + 1 - \frac{2^7}{x^2}\right) \sim 1 + x - 2^5 \cdot \left(\frac{2}{x}\right)^2 = \ldots = 1 + t_1^0 t_2^0 \cdot \overbrace{(t_1 - t_1^5 t_2^7)}^{\psi_p \in \mathfrak{m}_p - \mathfrak{m}_p^2}$$

[LL99] Qing Liu and Dino Lorenzini. "Models of Curves and Finite Covers". In: *Compositio Mathematica* 118.1 (1999), pp. 61–102. DOI: 10.1023/A:1001141725199.

[CE03] Brian Conrad and Bas Edixhoven. "$J_1(p)$ *Has Connected Fibers*". In: *Documenta Mathematica* 8 (Jan. 2003), pp. 331–408.

[Rüt14] Julian Rüth. "Models of Curves and Valuations". PhD thesis. Ulm University, 2014.

[Sri15] Padmavathi Srinivasan. *Conductors and minimal discriminants of hyperelliptic curves with rational Weierstrass points*. 2015. arXiv: 1508.05172 [math.AG]. URL: https://arxiv.org/abs/1508.05172.

[OS22] Andrew Obus and Padmavathi Srinivasan. *Explicit minimal embedded resolutions of divisors on models of the projective line*. 2022. arXiv: 2105.03030 [math.AG].