Transcendental Brauer group of certain K3 surfaces

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Introduction

Let X be a *nice* variety over $\mathbb{Q}$. The Brauer group of X is

$$\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m).$$

Number theoretical application:

$$\overline{X(\mathbb{Q})} \subset X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \subset X(\mathbb{A}_\mathbb{Q}) = \prod_{p \leq \infty} X(\mathbb{Q}_p)$$

If $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$, but $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \emptyset$ then BMO to Hasse principle.
Important subgroups:

$\mathrm{Br}_1(X) = \ker(\mathrm{Br}(X) \to \mathrm{Br}(\bar{X}))$	algebraic elements
$\mathrm{Br}(X)/\mathrm{Br}_1(X)$	transcendental elements

Construction of K3 family

Let $C : ax^3 + by^3 + cz^3 = 0 \subset \mathbb{P}^2_{\mathbb{Q}}$ and let $\rho \in Aut(C \times C)$ given by:

$$\rho((P, Q)) = (Q, R) \text{ with } P, Q, R \text{ collinear.}$$

Let $Y = (C \times C)/\rho$ be the quotient. It's a K3 surface!

Proposition (van Luijk, 2007)

If *abc* is not a cube in $\mathbb{Q}$, then $\mathrm{Br}_1(Y)$ is *trivial*.

Proposition (N. 2024+)

For every n, there is an injection

$$\mathrm{Br}(Y)_n / \mathrm{Br}_1(Y)_n \hookrightarrow \mathrm{Br}(C \times C)_n / \mathrm{Br}_1(C \times C)_n$$

which is an isomorphism if $(n, 3) = 1$.

Result

Theorem (N. 2024+)

If *abc* and $4abc$ are not cubes in $\mathbb{Q}$, then $\mathrm{Br}(Y) = \mathrm{Br}_1(Y)$, thus *trivial*.

Skorobogatov's conjecture states that BMO is the only obstruction to Hasse principle for K3 surfaces, therefore we conjecture

Conjecture

Any everywhere locally solvable curve $C : ax^3 + by^3 + cz^3 = 0$, with *abc* and $4abc$ not cubes in $\mathbb{Q}$, has a point in a Galois cubic extension.