A positive density of elliptic curves are diophantine stable in certain Galois extensions

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The Mordell conjecture 100 years later

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Diophantine stability

Let K be a number field and L be a finite extension of K.

If E is an elliptic curve defined over K, then $E(K) \subseteq E(L)$.

Definition

An elliptic curve $E_{/K}$ is said to be ***diophantine stable*** *in L, if $E(K) = E(L)$.*

Fix $K = \mathbb{Q}$ and a degree p Galois extension $L/\mathbb{Q}$ where $p \in \{3, 5\}$.

The elliptic curves $E_{/\mathbb{Q}}$ admit a short Weierstrass form $E = E_{A,B} : y^2 = x^3 + Ax + B$ for some $A, B \in \mathbb{Z}$ such that either $\ell^4 \nmid A$ or $\ell^6 \nmid B$ for all primes ℓ.

Definition

Define the ***naive height*** *$h(A, B)$ of $E_{A,B}$ by $h(A, B) := max\{|A|^3, B^2\}$.*

Density result

Let p be a prime in $\{3, 5\}$, $L/\mathbb{Q}$ be a Galois extension of degree p and Z be the set of primes $\ell \in \mathbb{Z}$ that are ramified in L.

Theorem (A. Ray, P. Shingavekar)

Let p, L and Z be as above. Assume that

1. $2, p \notin Z$,
2. *for all* $\ell \in Z$, *there exists an elliptic curve* $\mathbb{E}$ *over* $\mathbb{F}_\ell$, *such that* $\mathbb{E}(\mathbb{F}_\ell)[p] = 0$ *and* $\mathbb{E}^{-1}(\mathbb{F}_\ell)[p] = 0$. *Here,* $\mathbb{E}^{-1}$ *is the quadratic twist of* $\mathbb{E}$ *by* -1.

Then there is a positive density of elliptic curves $E_{/\mathbb{Q}}$, *when ordered by their naive height, such that*

$$E(L) = E(\mathbb{Q}) = 0.$$

Effective lower density

Definition

For $\ell \in Z$, let
$\mathfrak{a}_\ell := \{(\bar{A}, \bar{B}) \in (\mathbb{Z}/\ell\mathbb{Z})^2 \mid 4\bar{A}^3 + 27\bar{B}^2 \neq 0, E_{\bar{A},\bar{B}}(\mathbb{F}_\ell)[p] = 0 \text{ and } E_{\bar{A},-\bar{B}}(\mathbb{F}_\ell)[p] = 0\}$.

Theorem (A. Ray, P. Shingavekar)

The lower density of this set of elliptic curves is at least $\eta_p \prod_\ell \delta_\ell$, where

$$\eta_p := \begin{cases} \frac{1}{4}, & \text{if } p = 3; \\ \frac{3}{8}, & \text{if } p = 5; \end{cases} \quad \text{and} \quad \delta_\ell := \begin{cases} \frac{2}{3}, & \text{if } \ell = 3, \ell \notin Z \text{ and } p = 5; \\ 1 - \frac{2}{\ell^2} + \frac{1}{\ell^3}, & \text{if } \ell \notin Z \cup \{2, p\} \text{ and } \ell \neq 3; \\ 1 - \frac{1}{\ell}, & \text{if } \ell = p; \\ \frac{1}{2^{21}}, & \text{if } \ell = 2; \\ \frac{\#\mathfrak{a}_\ell}{\ell^2}, & \text{if } \ell \in Z. \end{cases}$$

Thank you for your attention!