Hilbert's 10^{th} Problem for some new families of # fields

Q (Hilbert, 1900) Is there an algorithm so that:

Input: $f(\vec{x}) \in R[x_1, \ldots, x_n]$, $R = \mathbb{Z}$

Output: $\begin{cases} \text{Yes} & \text{if } \exists\, \vec{t} \in R^n \text{ s.t. } f(\vec{t}) = 0 \\ \text{No} & \text{if not.} \end{cases}$

?

A (Matiyasevič, 1970) NO.

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(Conj. (Denef, Lipshitz, 1974)) There is no algorithm so that:

Input: $f(\vec{x}) \in R[x_1, \ldots, x_n]$, $R = \mathcal{O}_F$, F a # field

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Conj. known when $F = \mathbb{Q}(\sqrt[3]{n})$

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Thm (Poonen, Shlapentokh)

Suppose DLC holds for some # field F.

If $\exists$ elliptic curve E/F so that:

$\operatorname{rank} E(F) = \operatorname{rank} E(F') > 0$

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Idea (García-Fritz, Pasten) If DLC holds for F, find elliptic curve E so that:
① $\operatorname{rank} E(F) = 0$, ② $\operatorname{rank} E(\mathbb{Q}(\sqrt{d})) > 0$. By easy quadratic twist argument, $\operatorname{rk} E^{(d)}(F) = \operatorname{rk} E^{(d)}(F(\sqrt{d})) > 0$

So Poonen + Shlapentokh $\Rightarrow$ DLC for $F(\sqrt{d})$!

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Thm (García-Fritz, Pasten) DLC holds for $F = \mathbb{Q}(\sqrt[3]{p}, \sqrt{-q})$, where $p \in \mathcal{P}$, $q \in \mathcal{Q}$ sets of primes of (Čebotarev) densities $\delta(\mathcal{P}) = \frac{5}{16}$, $\delta(\mathcal{Q}) = \frac{1}{12}$

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DLC holds for $F = \mathbb{Q}(\sqrt[3]{p}, \sqrt{q \times 7})$, $p \in \mathcal{P}'$, $q \in \mathcal{Q}'$ primes.
$\delta(\mathcal{P}') = \frac{103}{128}$, $\delta(\mathcal{Q}') = \frac{1}{36}$

References:

García-Fritz, Pasten: Math. Annalen 377 (2020)

Kundu, Lei, S. : Math. Annalen doi: 10.1007/s00208-024-02879-9

Thank you!