A positive proportion of monic odd hyperelliptic curves have no unexpected quadratic points

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The Mordell conjecture 100 years later
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The case of rational points

- Let $\mathscr{F}_g$ be the family of monic odd hyperelliptic curves of genus $g \geq 2$

$$y^2 = x^{2g+1} + c_{2g-1}x^{2g-1} + \cdots + c_0, \quad \text{for } c_i \in \mathbb{Z}$$

- Falting's Theorem $\Longrightarrow \#C(\mathbb{Q}) < \infty$ for each $C \in \mathscr{F}_g$, but each curve in $\mathscr{F}_g$ has an "expected" $\mathbb{Q}$-rational Weierstrass point at ∞

Question

When curves $C \in \mathscr{F}_g$ are ordered by height ($:= \max\{|c_i|^{1/i}\}$), how often does C have no unexpected $\mathbb{Q}$-rational points?

Theorem (Poonen-Stoll, 2013)

When curves $C \in \mathscr{F}_g$ are ordered by height, the proportion such that $C(\mathbb{Q}) = \{\infty\}$ is $\gg 2^{-O(g^2)} > 0$ for every $g \geq 3$; and tends to 100% exponentially fast as $g \to \infty$.

The case of quadratic points

- Given $C \in \mathscr{F}_g$ and $P \in (\mathrm{Sym}^2 C)(\mathbb{Q})$, we call P *expected* if P is the preimage of $\mathbb{Q}$-rational point under the hyperelliptic map $C \to \mathbb{P}^1_{\mathbb{Q}}$
- Faltings proved that if $g \geq 4$, the set of unexpected points in $(\mathrm{Sym}^2 C)(\mathbb{Q})$ is finite

Question

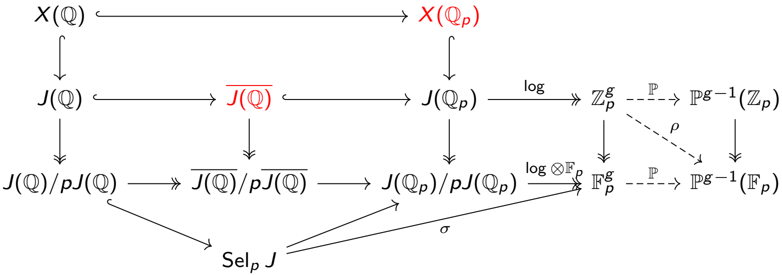
When curves $C \in \mathscr{F}_g$ are ordered by height, how often does $\mathrm{Sym}^2 C$ have no unexpected $\mathbb{Q}$-rational points? (I.e., how often does C have no unexpected quadratic points?)

Theorem (LS, 2024)

When curves $C \in \mathscr{F}_g$ are ordered by height, the proportion with the property that $\mathrm{Sym}^2 C$ has no unexpected $\mathbb{Q}$-rational points is $\gg 2^{-O(g^2)} > 0$ for every $g \geq 4$.

Method of proof: Selmer-group Chabauty

- Let C be monic odd hyperelliptic of genus $g \geq 4$. Let $J = \mathrm{J}(C)$ be the Jacobian, and let $X = \mathrm{im}(\mathrm{Sym}^2 C \to J)$
- For a prime p, let $\overline{J(\mathbb{Q})} \subset J(\mathbb{Q}_p)$ be the p-adic closure of $J(\mathbb{Q})$ in $J(\mathbb{Q}_p)$. Consider the following diagram:



- Use results of Bhargava-Gross on the equidistribution of 2-Selmer elements to deduce that $\mathbb{P}\sigma(\mathrm{Sel}_2 J) \cap \rho \log(X(\mathbb{Q}_2)) = \varnothing$ a positive proportion of the time