Effective bounds for Gabber's theorem

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Gabber '83

Let X be a smooth, projective variety (of dimension n) over a separably closed field k. Then the integral ℓ-adic cohomology $\mathrm{H}^i(X, \mathbb{Z}_\ell)$ are torsion free for all but finitely many ℓ.

Deligne '80

Let $X/\mathbb{F}_q$. Let $P_i(X, T) := \det(1 - F_q^\star T \mid \mathrm{H}^i(X, \mathbb{Q}_\ell))$. Then for a Lefschetz pencil $(X_t)_{t \in \mathbb{P}^1}$ of hyperplane sections on X, $P_{n-1}(X, T)$ is the lcm of all polynomials $f(T)$ such that $f(T)^{(r)} \mid P_{n-1}(X_t, T)$ over all $t \in \mathbb{F}_{q^r}$, with X_t smooth.

Theorem

There exists a polynomial function $\pi(x)$ such that for any smooth projective degree D variety $X \subset \mathbb{P}^N$, we have $\mathrm{H}^i(X, \mathbb{Z}_\ell)$ is torsion-free for $\ell > \pi(D^N)$.

Surface case

- Fibre X as a Lefschetz pencil $(X_t)_{t\in\mathbb{P}^1}$ of hyperplane sections, giving rise to a morphism $\pi : \tilde{X} \to \mathbb{P}^1$ whose fibres are X_t.
- Consider the sheaf $\mathcal{F} = R^1\pi_\star\mathbb{Z}_\ell$ on $\mathbb{P}^1$, locally constant on open $U \subset \mathbb{P}^1$.
- $\mathcal{E} \subset \mathcal{F}|_U$, subsehaf of integral ℓ-adic vanishing cycles. Hard Lefschetz $\implies$ $\mathcal{F}_u \otimes \mathbb{Q}_\ell \simeq H^1(X, \mathbb{Q}_\ell) \oplus (\mathcal{E}_u \otimes \mathbb{Q}_\ell)$ for $u \in U$. $P_1(X_u, T) = P_1(X, T)P_{\mathcal{E}_u}(T)$. Write $\mathcal{V} := \mathcal{E} \otimes \mathbb{F}_\ell$.

- Monodromy of $\mathcal{V}$ is 'big', i.e. the associated map $\pi_1(\overline{U}, \eta) \to \mathrm{Sp}(2r, \mathbb{F}_\ell)$ is surjective for $\ell > 5$, different from char.
  - Local monodromy contains a transvection coming from Picard-Lefschetz formulas.
  - Associated representation is irreducible. Idea: Katz's middle convolution algorithm.
  - Theorem of Hall $\implies$ image is full.
- Want to find $u_1, u_2 \in \mathbb{F}_Q$ from a bounded extension, and ℓ large enough, such that $P_{\mathcal{E}_{u_1}}(T), P_{\mathcal{E}_{u_2}}(T)$ are coprime mod ℓ.
- Analysis of the proportion of symplectic similitudes with char poly coprime to a given one gives bounds for ℓ.
- Error term in equidistribution of Frobenius gives bounds for Q for which distinct u_1, u_2 exist with above property.