

Computing genus 2 curves over \mathbb{Q} whose Jacobian has good reduction away from 2

The Mordell conjecture 100 years later, Massachusetts Institute of Technology

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8 July 2024

Genus 2 curves

Problem (Poonen 1996)

List all genus 2 curves C/\mathbb{Q} whose Jacobians have good reduction away from 2.

Smart (1997) computed all 366 genus 2 curves with good reduction outside 2. But there are more! Examples of other curves C/\mathbb{Q} where $\text{Jac}(C)$ good outside 2:

- $C : y^2 = x^5 - 14x^3 + 81x$ has bad reduction at $\{2, 3\}$.
- $C : y^2 = 2x^5 - 9x^4 - 24x^3 + 22x^2 + 78x - 41$ has bad reduction at $\{2, 5\}$.
- $C : y^2 = 2x^5 + x^4 - 16x^3 - 72x^2 + 240x + 136$ has bad reduction at $\{2, 7\}$.
- $C : y^2 = x^5 + 478x^3 + 57122x$ has bad reduction at $\{2, 13\}$.
- $C : y^2 = x^5 + 28x^4 - 868x^3 - 6160x^2 + 43076x - 149072$ has bad reduction at $\{2, 3, 11\}$.

So far, we've found 512 examples of genus 2 curves C/\mathbb{Q} such that $\text{Jac}(C)$ is good outside 2.

Genus 2 curves

Easier Problem

Fix a small set of primes S . Find all genus 2 curves C/\mathbb{Q} with good reduction outside S and whose Jacobian has good reduction away from 2.

Rough Idea:

- Let $C/\mathbb{Q} : y^2 = c(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)(x - \alpha_5)(x - \alpha_6)$ be such a curve, where $\alpha_i \in \mathbb{Q}(J[2])$.
- Let $\psi_1, \psi_2, \dots, \psi_t$ be a set of S -unit generators over $\mathbb{Q}(J[2])$.
For each pair $1 \leq i < j \leq 6$, let $\alpha_i - \alpha_j = \psi_1^{a_{1,i,j}} \psi_2^{a_{2,i,j}} \dots \psi_t^{a_{t,i,j}}$.
- Impose as many linear constraints on $a_{k,i,j}$ as we can: Galois constraints, cluster pictures for odd primes p , solutions to simple S -unit equations $\tau + \sigma(\tau) = 1 \dots$
- Solve the linear system! Brute force, integer programming, closest vector problem...

Summary

Theorem (V. WIP)

There are at least 512 \mathbb{Q} -isomorphism classes of genus 2 curves C/\mathbb{Q} whose Jacobian has good reduction away from 2. In particular,

- 1. There are exactly 78 such genus 2 curves C/\mathbb{Q} with $\text{rad}(\Delta_{\min}) = 6$.*
- 2. There are exactly 28 such genus 2 curves C/\mathbb{Q} with $\text{rad}(\Delta_{\min}) = 10$.*
- 3. There are exactly 24 such genus 2 curves C/\mathbb{Q} with $\text{rad}(\Delta_{\min}) = 14$.*

All such genus 2 curves C/\mathbb{Q} with $|\Delta_{\min}| \leq 10^{14}$ is contained in our table.

- All curves (and more stats) given at: bit.ly/genus2
- Are there any curves C/\mathbb{Q} with $\text{Jac}(C)$ good outside 2 not in our list?
- If you can find any more curves, you'll win £100.¹

¹Terms and conditions apply!