Models of CM Elliptic Curves with Prescribed ℓ-adic Galois Image

Benjamin York

joint with Enrique González-Jiménez & Álvaro Lozano-Robledo

University of Connecticut

July 9th, 2024

Question

For fixed F and prime ℓ, what are the possible images (up to conjugation) of ρ_{E,ℓ^∞} in $\mathrm{GL}(2,\mathbb{Z}_\ell)$ as E/F varies?

Mazur's "Program B": (from "Rational points on modular curves", 1976-77)

B. Given a number field K and a subgroup H of $GL_2\hat{\mathbb{Z}} = \prod_p GL_2\mathbb{Z}_p$ classify all elliptic curves $E_{/K}$ whose associated Galois representation on torsion points maps $\mathrm{Gal}(\bar{K}/K)$ into $H \subset GL_2\hat{\mathbb{Z}}$.

Question (An Inverse to Mazur's Program)

For a prime ℓ and subgroup $H \subseteq \mathrm{GL}(2,\mathbb{Z}_\ell)$, if H is a possible Galois image for an elliptic curve, for what curves E do we have that $\mathrm{Im}\,\rho_{E,\ell^\infty}$ is conjugate to H?

Main Result

Theorem (González-Jiménez, Lozano-Robledo, Y, 2023)

Let $\mathcal{O} = \mathcal{O}_{K,f}$ be an order of imaginary quadratic field K with conductor f, and suppose $\mathcal{O}$ has class number 1 or 2. Then

1. *We give a Weierstrass model for an elliptic curve $E = E_{\mathcal{O}}$, defined over $\mathbb{Q}(j_{K,f})$, such that E has CM by $\mathcal{O}$. We choose this curve to have minimal conductor norm among all curves with CM by $\mathcal{O}$.*
2. *We give a description of all possible ℓ-adic Galois images of elliptic curves with CM by $\mathcal{O}$. Further, given a possible ℓ-adic Galois image $H \subseteq \mathrm{GL}(2, \mathbb{Z}_\ell)$, we describe all twists of $E_{\mathcal{O}}$ whose ℓ-adic image is conjugate to H.*

CM 2-adic Galois Images $\Delta_K = -3$ and $h(\mathcal{O}_{K,f}) = 1, 2$

Δ_K	f	$\mathbb{Q}(j_{K,f})$	$G_{E,2^\infty}$	twist	conditions
-3	1	$\mathbb{Q}$	$\mathcal{N}_{-1,1}(2^\infty)$	t	$t \in \mathbb{Q}^*,\ \ t \notin 4\mathbb{Q}^3$
			$\langle \mathcal{C}_{-1,1}(2^\infty)^3, c_1' \rangle$	$4t^3$	$t \in \mathbb{Q}^*$
	2	$\mathbb{Q}$	$\mathcal{N}_{-3,0}(2^\infty)$	t	$t \in \mathbb{Q}^*$
	3	$\mathbb{Q}$	$\mathcal{N}_{-9,3}(2^\infty)$	t	$t \in \mathbb{Q}^*$
	4	$\mathbb{Q}(\sqrt{3})$	$\mathcal{N}_{-12,0}(2^\infty)$	t	$t \in \mathbb{Q}(\sqrt{3})^*,\ \ t \neq \pm 1, \pm 2$
			$\langle G_{-12,0}^{2,1}, c_1 \rangle$	-2	2.2.12.1-256.1-c4
			$\langle G_{-12,0}^{2,2}, c_1 \rangle$	-1	2.2.12.1-16.1-a4
			$\langle G_{-12,0}^{2,2}, c_{-1} \rangle$	1	2.2.12.1-16.1-a3
			$\langle G_{-12,0}^{2,1}, c_{-1} \rangle$	2	2.2.12.1-256.1-c3
	5	$\mathbb{Q}(\sqrt{5})$	$\mathcal{N}_{-25,5}(2^\infty)$	t	$t \in \mathbb{Q}(\sqrt{5})^*$
	7	$\mathbb{Q}(\sqrt{21})$	$\mathcal{N}_{-49,7}(2^\infty)$	t	$t \in \mathbb{Q}(\sqrt{21})^*$