

Algebraic Independence of Special Points on Shimura Varieties

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Previous Results

Definition

Let E be an elliptic curve over $\overline{\mathbb{Q}}$, a subgroup $\Gamma \subset E(\overline{\mathbb{Q}})$ is *finite-rank* if $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite-dimensional.

Theorem (Buium–Poonen ('07), Pila–Tsimmerman ('19))

Let $S = X(N)$ be the modular curve of level N , $\Sigma \subset S$ be the CM-points of S , and $\Gamma \subset E(\overline{\mathbb{Q}})$ be a finite-rank subgroup. Let $\Phi: S \rightarrow E$ be a dominant morphism, then $\Phi(\Sigma) \cap \Gamma$ is finite.

Theorem (Pila–Tsimmerman, '14)

Identify the CM-points Σ of a modular curve with their j -invariants, $\Sigma \subset \mathbb{C}$. Let $\Gamma \subset \mathbb{C}^\times$ be a finite-rank subgroup. Then $\Sigma \cap \Gamma$ is finite.

Our Result

Theorem (Fu-Z. ('24))

Let G be a commutative algebraic group defined over a field of characteristic 0, and suppose S is a Shimura variety with simple Shimura datum. Let $\Sigma \subset S$ be the special points of S and $\Gamma \subset G$ be a finite-rank subgroup. Let $\Phi: S \rightarrow G$ be a dominant morphism. There exists a proper subvariety $V \subset S$ such that $\Sigma \cap \Phi^{-1}(\Gamma) \subset V$.