Towards a model theory of heights

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I. Two trichotomies

Manin's talk, Novosibirsk 1989. Topological trichotomy: genus 0, genus 1, higher genus. Reflected in arithmetic. Gödel.

Let X be a set defined by some formulas in a structure M. We say X is *minimal* if it is infinite, but cannot be definably split into two infinite sets; and this is *uniform*, i.e. for every formula $\phi(x, y)$, for some n, for all b $|\{x \in X : \phi(x, b)\}|$ has at most m points, or all but m points. (For $M = (\mathbb{C}, +, \cdot)$, minimal sets curves ($\pm$ finite sets.)
An *algebraic function* (definable in M) from Y to X is a definable subset of $Y \times X$, whose projection to Y is onto and $\leq m$-to-one.
The relation: $x \in \mathrm{acl}(x_1, \ldots, x_n)$ has the properties of a pre-geometry; it thus gives rise to a dimension theory on definable subsets of X^n.

Zilber's trichotomy

A classification of minimal sets:
Trivial geometry. No definable families of irreducible subsets of X^n, other than ones like $X^{n-1} \times \{b\}$.
Locally modular. No high-dimensional families of irreducible subsets of X^n.
In this case one can prove existence of an abelian group A, isogenous to X, such that every definable subset of A^n is a finite Boolean combination of cosets of definable subgroups.
Field-like: high-dimensional families exist.
We say *Zilber's conjecture holds* (in M) if every field-like minimal X is isogenous to an algebraically closed field.

The need for intermediaries

A first match : given a curve C over $\mathbb{C}$, consider the structure consisting of all rational maps $C^n \to C$.
Genus 0 $\leftrightarrow$ fieldlike,
genus 1 $\leftrightarrow$ locally modular,
higher genus $\leftrightarrow$ trivial alg. closure geometry.
However not much arithmetic structure is visible in this interpretation.
We could try to simply take $\mathbb{Q}$-points in place of $\mathbb{C}$-points; but this begs the question in genus ≥ 1, and blows up (Gödel, J. Robinson) in genus 0.
To make a real connection, need intermediate theories that see more of the diophantine geometry, but are model-theoretically tractable.

Differential fields.

Char. $p > 0$ equivalent to separably closed fields with $[K : K^p] = p$.
Axiomatizability of existentially closed differential fields (A. Robinson, Blum)
Likewise separably closed fields in char. $p > 0$ (with $[K : K^p] = p$.) (Ershov, Delon).

Minimal sets exist; Zilber's conjecture holds.

An abelian variety A over a differential field K admits a definable map $m : A \to K^n$ into a vector group. (Picard-Fuchs, Manin, Buium in char. 0). The *Manin kernel* $A_0 = m^{-1}(0)$ is finite-dimensional. Hence any finitely generated subgroup Γ of A is contained in a finite-dimensional *definable* subgroup $\widetilde{\Gamma}$ of A (namely $m^{-1}(V)$, where V is the k-space generated by $m(\Gamma)$, $k =$ the constant field.).

Structure of the Manin kernel and Mordell-Lang for function fields

Let A be an Abelian variety with zero trace to the constant field.
A_0 the Manin kernel.
Let $Y \subset A$ be minimal (cut out by some differential equations).
The geometry of Y cannot be trivial; this is guaranteed by the group structure.
If Y is field-like, it is isogenous to a definable field. But it can be shown that the only field of finite dimension interpretable by differential equations is the field of constants. This leads to Y being isotrivial.

Hence Y is locally modular. With a little further analysis one shows A_0 is locally modular.
This already shows that for any subvariety X of A, $X \cap A_0$ is a coset of a group.

An additional idea associated with the global effect of minimal sets, is needed to bridge the gap between $\widetilde{\Gamma}$ and A_0. Analogy with Poonen's Mordell-Lang + Bogomolov.
Mordell-Lang for function fields follows.
En route we saw that The nontrivial minimal sets, up to isogeny, are just the Manin kernels of simple, non-isotrivial abelian variety, and the field of constants.

Further results (over number fields, and for Drinfeld modules by Scanlon) use difference fields as an intermediary instead. This requires also an extension of stability theory and the Zilber trichotomy (Chatzidakis-H.)
Could there be a model-theoretically tame theory capturing (part of) the geometry of heights?

II. The theory GVF

Three presentations (from Ben-Yaacov, H., Destic, Szachniewicz, soon on ArXiv.)

1) Direct axiomatization via Weil heights:

A GVF is a field F along with $Sym(n)$-invariant real-valued functions $h : \mathbb{P}^n(F) \to \mathbb{R}^{\geq 0}$ such that:

- $h((x_1 : \ldots : x_n)) \leq h((x_0 : \ldots : x_n))$, with equality if $x_0 = 0$.
- $h((x_i y_j)_{i,j}) = h((x_i)) + h((y_j))$, and
- $h((x_1 + y_1, \ldots, x_n + y_n)) \leq h(x_1, \ldots, x_n, y_1, \ldots, y_n) + e$. When $e = 0$, we talk of *GVFs of function field type.*
- Write $ht(a) = h(a : 1)$. Then $ht(1) = 0$

Aside: You may have noticed these are real-valued formulas, whereas traditionally logical formulas can take two values. Surprisingly little adjustment is needed.

Additional *definable* $\mathbb{R}$-*valued functions* are formed by closing under $+, \cdot$, $\inf, \sup$ and uniform limits. In case $\inf, \sup$ are not used, we talk of *qf-definable* functions.

A set of the form $X = \{x : f(x) = 0\}$ will be called *closed* or $\bigwedge$-*definable.*

X is *minimal* if (in some model) it is infinite, while (in any model of the axioms) any formula $\phi(x)$ takes a unique generic value α on X; any other value is taken only finitely often.

All the basic results of logic (notably compactness) remain valid.

Alternative presentations

2. More general local terms.
Usual field operations $+, -, \cdot$ and relations $=, \neq$ on F.
A symbol R_t for each continuous, positively homogeneous function on $\mathbb{R}^n$.
Intended intepretation over a number field:

$$R_t(x_1, \ldots, x_n) = \sum_p t(v_p x_1, \ldots, v_p x_n)$$

where the v_p are the valuations and $-\log$ absolute values of the field, normalized so that the product formula holds: $\sum_p v_p(x) = 0$.

Axioms for R_t presentation

1. $(F, +, \cdot)$ is an integral domain.

2. $\{R_t\}$ Compatible with permutations of variables, dummy variables.

3. (Linearity:) $R_{t_1+t_2} = R_{t_1} + R_{t_2}$. $R_{\alpha t} = \alpha R_t$.

4. (Positivity) For an affine variety $X \subset \mathbb{A}^n$: If $t(v(x)) \geq 0$ for every absolute value and every $x \in X$, then $R_t(a) \geq 0$ for $a \in X$.

5. (Product formula) $R_{Id}(x) = 0$ for $x \neq 0$.

Weil heights are given by the special case $t(u_0, \ldots, u_n) = \min(u_1, \ldots, u_n)$. They suffice to capture all R_t. But the R_t presentation makes it clear that we have not lost sight of local data. For instance, employing $\max(-v(2), 0)$ which vanishes on non-archimdean v, we can probe the measure on archimedean valuations / embeddings into $\mathbb{C}$, likewise $\mathbb{Q}_p$..

3. Measure theoretic presentation (cf. Gubler's M-fields, Chen-Moriwaki adelic curves) For any (countable) $K \models GVF$, there exists a measure μ on the space of absolute values of K, $v(x) = -\log|x|$, such that $(v \mapsto v(a))$ is in $L^1(\mu)$, and

$$R_t(x_1, \ldots, x_n) = \int t(v(x_1), \ldots, v(x_n)) d\mu(v)$$

μ is unique up to a suitable renormalization (v with mass m) $\rightsquigarrow$ ($2v$ with mass $m/2$.)

A word about ultraproducts or logical compactness

Theorem (Gödel, Skolem, Malcev, Tarski, A. Robinson, Łoś). *Given a sequence M_i of structures for some language, one can find a subsequence M_j and a 'limit' M such that a sentence is true in M iff it is true in almost every M_j.*
In the continous version, where a formula takes real values, $\phi^{M_j} \to \phi^M$.

Examples:
1) ($L = \{+, \cdot\}$). Lefschetz principle - passage from large char. $p > 0$ to char 0 or back - as used by e.g. Ax,Deligne, Mori, Kontsevich...
2) $\mathbb{F}_p(p = 2, 3, \cdots) \to F$, F a field of characteristic 0, with exactly one finite extension of any degree n.

3) L = language of GVFs. $M_i = \overline{\mathbb{Q}}[1/i]$, a renormalization of the GVF $\overline{\mathbb{Q}}$; use the standard Weil heights divided by i. Then the limit M is a GVF *of function field type*. The set of elements of height 0 form a subfield, the constant field, containing $\overline{\mathbb{Q}}$. We have $\overline{k(t)} \leq M$, where $ht(k) = 0$ and $ht(t) = 1$.
(Note e.g. the archimedean places have measure $1/i$ in $\mathbb{Q}[1/i]$, hence measure 0 in M , as do the 2-adic, 3-adic, ...)
Axiomatization has *uniformity, effectiveness, transfer* as standard consquences. If a statement is true *for every model of the axioms* and not only the one we are interested in, then it must be true uniformly in parameters. Thus Tarski proved polynomial bounds on distance from a singularity in the prehistory of o-minimality.)
For GVFs this makes it possible to transfer some statements from function fields to number fields. Let us see this for a version of the "gap principle" from the talks of Gao, Kühne and Yuan.

Transfer principle, uniformity

We take a version of the 'gap principle' as an example for both transfer (function fields to number fields) and uniformity.

Let A be an Abelian variety over a GVF K. There exists a unique maximal $\bigwedge$-definable subgroup of bounded height; the *Bogomolov kernel* A_0 of A. For any symmetric ample line bundle L on A, A_0 can be defined by $\hat{h}_L(x) = 0$.
We say A is isotrivial if $k := \{x \in K : ht(x : 1) = 0\}$ is a subfield of k, and A descends to k.
We say that the Bogomolov conjecture holds over a GVF K if for any non-isotrivial Abelian variety A/K, for subvariety X of A containing no positive-dimensional translate of a subgroup of A, no contained in a proper coset, $X \cap A_0$ is finite.

Take a family $(A(s), L(s)) : s \in S)$ of Abelian varieties equipped with ample symmetric line bundle; parameterized by some constructible $S \subset \mathbb{P}^n$. Let $\hat{h}$ denote the corresponding canonical height on any $A(s)$. ht just denotes Weil height on projective space.
Also consider a subvariety $X(t)$, parameterized non-redundantly by some $T \to S$. Assume for simplicity that $X(t)$ is a curve of genus > 1.

Proposition. *(1) implies (2):*

1. *The Bogomolov conjecture holds over every GVF of char.* 0*.*
2. *(Gap principle) There exist positive constants c_1, c_2 such that for any $s \in S(\overline{\mathbb{Q}})$ and $t \in T(\overline{\mathbb{Q}})$ above s, and $P \in A(\overline{\mathbb{Q}})$,*

$$\sharp\{Q \in X_t(\overline{\mathbb{Q}}) : \hat{h}(Q - P) \leq c_1 ht(s)\} \leq c_2$$

Proof. Replacing Q by $Q - P$ and the family $\{X_t : t\}$ by $\{X_t - P : (t, P) \in T \times A\}$ (made non-redundant), we reduce to the case $P = 0$.

Suppose (2) fails, and choose s_n, t_n with

$$\sharp\{Q \in X_{t_n}(\overline{\mathbb{Q}}) : \hat{h}(Q) \leq ht(s_n)/n\} > n$$

Choose $Q_{n,1}, \ldots, Q_{n,n}$ demontrating this. Let

$$(K, s, t, Q_1, Q_2, \ldots)$$

be a limit of

$$(\mathbb{Q}[1/ht(t)], s_n, t_n, Q_{n,1}, \ldots, Q_{n,n}, 0, 0, \ldots)$$

$A := A_s$, $X := X_t$. Note $ht_K(t) = 1$. Also, $\hat{h}(Q_i) \leq ht(s)/n$ for each n, so $\hat{h}(Q_i) = 0$. Thus each $Q_i \in X \cap A_0$. By (1), A is isotrivial. Since the Q_i are Zariski dense in X, X is also isotrivial; so $t \in T(k)$, contradicting $ht(t) = 1$.

$\square$

On the other hand, existential closedness of $\overline{k(t)}$ will imply that (1) for function-field type GVFS of the same characteristic as k is equivalent to uniform Bogomolov over $k(t)$. And a similar statement for $\overline{\mathbb{Q}}$. Thus for this application, the bridge is used in the opposite direction! But the point here is the general bridge itself.

Existentially closed GVFs

K is *existentially closed* if any solution to a GVF formula in a GVF extension of K, has an approximate solution in K.

Problem 0.1. *Is the class of existentially closed GVFs axiomatizable?*

If so, an arbitrary formula will be equivalent to an ‘existential’ one, $\inf_y |\phi(x,y)| = 0$, with ϕ q.f.
As we do not yet have an answer, we need to work at the level of qf formulas (qf- stability, qf-minimality.)

N.B. unlike the case of local fields, somewhat like the case of $\mathbb{R}_{an}$, there are many GVF-qf-algebraic functions that are not algebraic in the usual sense. An example, believed to generate them all, is the Minkowski smallest lattice vector in an adelic guise (Harder-Narasimhan canonical slope filtration of normed vector space over a GVF K.)

Theorem 1 (Szachniewicz). *$\bar{Q}$ is existentially closed as a globally valued field.*

Michał Szachniewicz, *existential closedness of $\bar{Q}$ as a globally valued field via Arakelov geometry*, ArXiv 2306.06275
A similar theorem for the function-field type GVF $\overline{k(t)}$ is proved in notes by Ben Yaacov-H, [GVF2], Arxiv.
Michał's theorem uses in particular recent results of Wilms. The function field version uses an extension of results of Boucksom-Demailly-Paun-Peternell and Boucksom-Favre-Jonsson.

Let's look at some soft corollaries.

1. Effectiveness given finiteness

Let X be variety over $\mathbb{Q}$, and consider algebraic solutions.
If one bounds the degree but not the height, *even if one knows that the number of solutions of degree $\leq d$ is finite*, there is still no known

algorithm to find them effectively.(Is this *provable)*?On the other hand if the height but not the degree is bounded:

Corollary 1. *Assume as known that X has only finitely many solutions in $\bar{Q}$ of height $< h_0$. Then these solutions can be computed effectively.*

Proof. If $\bar{Q} \leq L$ is any GVF extension, all solutions of X of height $< h_0$ must lie in $\bar{Q}$. Otherwise, one can recursively find infinitely many solutions in $\bar{Q}$: having found $a_1, \dots, a_n$, find a new solution b in some GVF extension; by the e.c. theorem, a solution b' exists in $\bar{Q}$ with $b' \notin \{a_1, \dots, a_n\}$ and $ht(b')$ as close as we like to $ht(b)$, hence $< h_0$.
In particular the number of solutions is bounded in any GVF.
Search for algebraic solutions $s_1, \dots, s_n$ and a formal proof from the GVF axioms that these are all the elements of X of height at most h_0. This is guaranteed to terminate, and in particular identify $s_1, \dots, s_n$.

□

Corollary 1 also works with additional qf GVF constraints, e.g. canonical height bounds, local conditions.

Corollary 2. *Every GVF is a subfield (with induced height structure) of an ultrapower of* $\overline{k(t)}$ *for some* k*, or of* $\overline{\mathbb{Q}}$*.*

Corollary 3. *A finiteness statement (e.g. Bogomolov) is true in every GVF extension of* $\overline{\mathbb{Q}}$ *if* and only if *it is true in* $\overline{\mathbb{Q}}$ uniformly in parameters.

Corollary 4. *Using Theorem 1, a sharp "statistical Fekete-Szego criterion" becomes available. Same Chebyshev number criterion as for the topological approximations.*

To illustrate, we give a simple statement in the 'forward' direction.

A statistical Fekete-Szego lemma

Lemma. *Let a_i be an infinite sequence of algebraic integers of bounded height, A_i the Galois orbit of a_i. Let C be a compact subset of $\mathbb{C}$. Assume: for any $\epsilon > 0$ and any open neighborhood U of C, for large enough i,*

$$\frac{|A_i \cap U|}{A_i} > 1 - \epsilon.$$

Then C has capacity ≥ 1, i.e. there exists a probability measure μ on X such that $\int \log(|x-y|) d\mu(x) d\mu(y) \geq 0$.

Proof: Let (K, a) be an ultraproduct of $(\overline{\mathbb{Q}}, a_i)$. The measure μ associated with the extension $\mathbb{Q} \leq \mathbb{Q}(a)$ for the real place must concentrate on C. For each p-adic place, the measure concentrates on $\mathbb{Z}_p$.
Let (K^*, a, a') be an ultraproduct of (K, a, a_i). The measure on $\mathbb{A}^2$ associated with $\mathbb{Q}(a, a')/\mathbb{Q}$ is the product measure $\mu \times \mu$, at each

place. Let $b = a - a'$. The measure associated with $\mathbb{Q}(b)$ satisfies $\int v(b)dv = 0$, and for each p-adic place the integral is ≥ 0; hence at the real place $\int -\log|b|_v dv \leq 0$, so $\int \log(x-y)d\mu(x)d\mu(y) \geq 0$.

Structure of GVF extensions

A GVF extension K/F *lives on* a normal variety X if $K = F(X)$, and the associated measure on $Val(K/F)$ concentrates on divisors of X.

Any GVF extension L/F of alt. closed fields is a direct limit of such, with decreasing ample heights; moreover approaching L uniformly in families of blowups.
Let $alb : X \to A$ be the Albanese variety of X, $J = Pic^0(X)$ the dual abelian variety, $\mathcal{P} \leq A \times J$ the Poincaré divisor. Let $\hat{h}_{\mathcal{P}}$ be the canonical height with respect to $\mathcal{P}$; it is bilinear, and $\hat{h}_{\mathcal{P}}(c, x)$ defines a bounded linear functional on H, the completion of $J(K)$ with respect to an ample canonical height on J. Hence $h_{\mathcal{P}}(c, x) = (x, nw)$ for a unique $nw \in H$.

Let K be a GVF, for simplicity with discrete GVF measure μ_K. Let $D_1, \ldots, D_{\dim NS(X)}$ be very ample divisors on X generating $NS(X) \otimes \mathbb{Q}$.

Proposition. *Let $L = K(c)$ be a finitely generated GVF extension, living on X. Then L determines, and is determined by:*

1. *A probability measure on $X_v(\mathbb{C})$ or on the Berkovich space X_v for each $v \in Supp(\mu_K)$.*
2. *$ht_{D_i}(c)$ for $i = 1, \ldots, \dim NS(X)$.*
3. *The Néron-Weil character $nw(L/K)$.*

We already see that Abelian varieties play a fundamental role within GVF formulas.

III. Minimal types

Theorem 2. *GVF is stable for qf formulas.*

Uses structure of GVF extensions, along with a transcendence style lema on existence of low degree polynoials vanishing on all generalized codimension 2 diagonals of a high power of X, joint with Ben Yaacov and Adiprasito.

One definition is that every qf type is definable. For instance, any sequence $(a_i) \in \overline{\mathbb{Q}}^n$ has a subsequence such that for any $b \in \overline{\mathbb{Q}}^m$, $ht(a_i, b)$ converges to $\delta(b)$, δ a GVF definable function.

Corollary. *Any infinite $\bigwedge$-definable set contains a qf minimal one.*

Problem 0.2. *Structure of minimal (qf) types?*

Conjecture 0.3. • *Zilber's conjecture holds for minimal types of GVF.*

- *The nontrivial qf-minimal types are (at least weakly) minimal.*

- *Every nontrivial qf-minimal type admits an isogeny to the Bogomolov kernel A_0 of a non-isotrivial simple abelian variety A, or $A = G_m$.*

- *A_0 is field-like only when $A = G_m$ and $ht(2) = 0$.*

In particular, for a non-isotrivial simple abelian variety A, or $A = G_m$ over a number field, minimality along with local modularity implies that A_0 is a pure module over $End(A)$, perhaps enriched with some constants.
qf-minimality of A_0 includes Zariski minimality (Bogomolov's conjecture) and adelic minimality (equidistribution). Many cases are proved by Bilu, Chambert-Loir, Szpiro-Ullmo-

Zhang, Zhang, Autissier, Gubler, Yamaki, Gao-Habbegger, Cantat-Gao-Habegger-Xie. Xie-Yuan, Chen-Moriwaki.
In addition qf-minimality includes the GVF living on A, ("no codimension two mass"); this would give at least weak qf minimality. Strict qf minimality includes finally determination of $ht_{D_i}(c)$ and the Néron-Weil character. This last vanishes by Cauchy-Schwarz $(a, c) \leq ||a||||c||$, and similarly the canonical heights $\widehat{ht}_{D_i}(c)$.
The further hypothesis of minimality for existential formulas, beyond qf-minimality, includes an amalgamation statement. Only in the case of $A = G_m$ over function fields is there any evidence for this; in this case A_0 is minimal with respect to étale projections of qf formulas too.
Could the strategy of the Manin kernel case apply here?

A step beyond qf stability

Proposition. *For $A = G_m$, over a function-field type GF, A_0 is minimal with respect to étale projections of qf formulas too.*