Rationality in Arithmetic Dynamics

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Notation

K is a number field unless otherwise specified

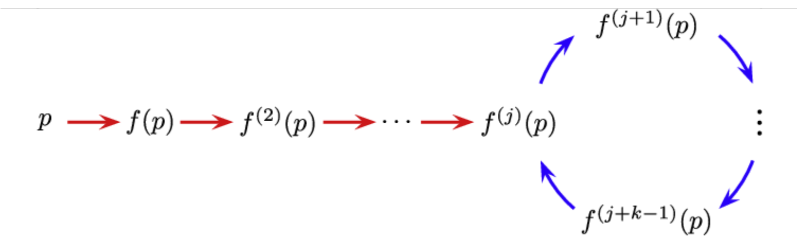
$f : \mathbb{P}^N_{\bar{K}} \to \mathbb{P}^N_{\bar{K}}$ is a morphism of degree $d \geqslant 2$ ($N = 1$ case will be focus)

$f^n := \underbrace{f \circ f \cdots \circ f}_{n \text{ times}}$

Let $\mathrm{Preper}(f, K) = \{P \in \mathbb{P}^N(K) \text{ preperiodic under f}\}$.

A preperiodic point $P \in \mathbb{P}^N(\bar{K})$ is one such that

$$|\{f^i(P)\}_{i=0}^{\infty}| < \infty.$$



Forward Orbits

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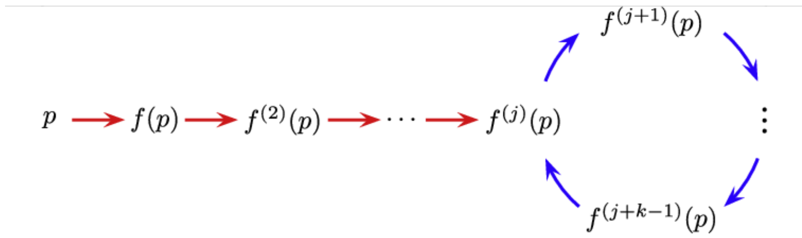


The *canonical height* of $P \in \mathbb{P}^N(\overline{K})$ is

$$\hat{h}_f(P) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(P)).$$

$$\hat{h}_f(P) = 0 \iff P \text{ is preperiodic under } f$$

Forward Orbits

We'll focus on two kinds of instances of K-rationality:

1. K-rationality of points of small canonical height (especially preperiodic points)
2. K-rational points on higher genus curves and their connection to forward orbits

Uniform Boundedness Principles

Let K be any field, and let $\mathcal{F}$ be a family of morphisms $f : \mathbb{P}^N_{\bar{K}} \to \mathbb{P}^N_{\bar{K}}$.

- We say that $\mathcal{F}$ satisfies the **Uniform Boundedness Principle** (UBP) over K if there is an $A = A(\mathcal{F}, K)$ such that for any $f \in \mathcal{F}(K)$,

$$|\text{Preper}(f, K)| \leqslant A.$$

- We say that $\mathcal{F}$ satisfies the **Strong Uniform Boundedness Principle** (SUBP) over K if for every $D \geqslant 1$ there is a $B = B(\mathcal{F}, D)$ such that for any extension L/K of degree $\leqslant D$ and any $f \in \mathcal{F}(L)$,

$$|\text{Preper}(f, L)| \leqslant B.$$

Examples

- Mazur–Kamienny–Merel: Lattès maps over $\bar{\mathbb{Q}}$ (degree > 1 maps on $\mathbb{P}^1_{\bar{\mathbb{Q}}}$ descended from endomorphisms on elliptic curves over $\bar{\mathbb{Q}}$) satisfy the SUBP over $\mathbb{Q}$

- Doyle–Poonen (2020): For k a field, $K = k(t)$, and $d \geqslant 2$ with $\text{char}(k) \nmid d$,

$$\mathcal{F} = \{z^d + c : c \in \bar{K} \backslash \bar{k}\}$$

satisfies the SUBP over K.

Another example

Taking a trivial family $\mathcal{F} = \{f\}$ addresses Northcott and Bogomolov-style results.

- Dvornicich–Zannier (2007): If K is a number field, and $f \in K[z]$ is any polynomial of degree $d \geqslant 2$ not conjugate to $\pm z^d$ or $T_d(\pm z)$, where T_d is the dth Chebyshev polynomial, then f has only finitely many preperiodic points in K^{cyc}, the maximal cyclotomic extension of K.

  In other words, $\mathcal{F} = \{f\}$ satisfies the UBP over K^{cyc}.

Uniform Boundedness Conjecture

Uniform Boundedness Conjecture (Morton–Silverman, 1994)

Let $N \geqslant 1$, let $d \geqslant 2$, and let K be a number field. Let $f : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be a degree d morphism defined over K. There is a $B = B(N, d, [K : \mathbb{Q}])$ such that $|\text{Preper}(f, K)| \leqslant B$.

Uniform Boundedness Conjecture restated

Let K, N, d be as above. The family $\mathcal{F}$ of degree d morphisms $\mathbb{P}^N_{\bar{K}} \to \mathbb{P}^N_{\bar{K}}$ satisfies the SUBP over $\mathbb{Q}$.

Example Uniform Boundedness Results

Theorem 1 (L., 2021)

Assume the *abcd*-conjecture. Let:

- K be a number field
- $d \geqslant 2$
- $\mathcal{F}$ be the set of degree d polynomials defined over K

Then $\mathcal{F}$ satisfies the UBP over K.

(A char. 0 function field analogue holds too.)

Abcd is a generalization of the *abc*-conjecture.

Two-Step Summary of Proof of Theorem 1

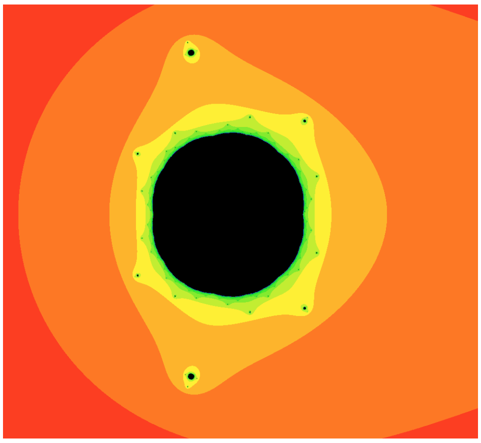
Step 1: Use the geometry of preperiodic points in the v-adic filled Julia sets to deduce arithmetic information about typical **pairwise differences** of preperiodic points.

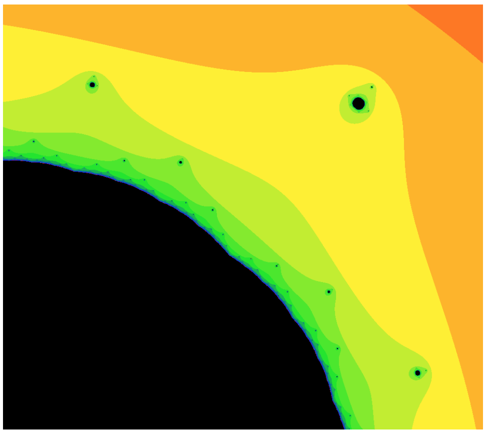
Step 2: Use arithmetic info about pairwise differences to derive a contradiction of *abc* or *abcd* if too many of these differences lie in K.

Example Uniform Boundedness Results

Theorem 2 (L., 2021)

Let $\mathcal{F} = \{f\}$, where $f \in K[x]$ is a polynomial with a periodic critical point $\neq \infty$ and at least one place of bad reduction.* Then $\mathcal{F}$ satisfies the UBP over K^{ab}.





* bad reduction here means not potentially good reduction

abc and uniform boundedness

A useful prototype to start with is an analogue in the Diophantine setting.

Theorem (Hindry–Silverman, 1988)

Assume the *abc*-conjecture. (An unconditional analogue holds over one-dimensional function fields of char. 0.)

Let:

- E/K be an elliptic curve with j-invariant j_E
- $\widehat{h}_E(P)$ be the Néron-Tate height of a K-rational point $P \in E(K)$.

Then there are explicit constants $c = c(K) > 0$ and $N = N(K)$, independent of E, such that there are at most N points $P \in E(K)$ satisfying

$$\widehat{h}_E(P) \leqslant c \max\{h(j_E), 1\}.$$

Outline of Hindry–Silverman

1. *abc* matters through the use of Szpiro's conjecture: given $\epsilon > 0$, there is a constant $c = c(K, \epsilon)$ such that

$$\log N_{K/\mathbb{Q}} \mathscr{D}_{E/K} \leqslant (6+\epsilon) \log N_{K/\mathbb{Q}} \mathscr{F}_{E/K} + c,$$

where $\mathscr{D}_{E/K}$ is the minimal discriminant and $\mathscr{F}_{E/K}$ is the conductor of E/K.

In other words, the valuations of $\mathscr{D}_{E/K}$ should not be too large on average.

Outline of Hindry–Silverman

2. For non-archimedean places v, there are two cases:
   - $|j_E|_v \leqslant 1$ (i.e., potential good reduction at v)
   - $|j_E|_v > 1$. Tate uniformization gives maps

$$\begin{array}{ccccc} E(K_v) & \xrightarrow{\sim} & K_v^\times / q^{\mathbb{Z}} & \longrightarrow & \mathbb{R}/(\log |j_E|_v \mathbb{Z}) \\ & & u & \longrightarrow & \log |u|_v \end{array}$$

   where $q \in K_v$ with $|q|_v = |1/j_E|_v < 1$.

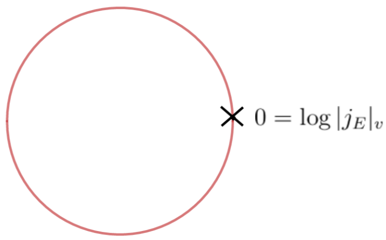
Outline of Hindry–Silverman

Two cases:

- $|j_E|_v \leqslant 1$ (i.e., potential good reduction at v)
- $|j_E|_v > 1$. Assuming for simplicity that E has ss reduction, Tate uniformization gives maps

$$E(K_v) \xrightarrow{\sim} K_v^\times/q^{\mathbb{Z}} \longrightarrow \mathbb{R}/(\log|j_E|_v\mathbb{Z})$$
$$u \longrightarrow \log|u|_v$$

where $q \in K_v$ with $|q|_v = |1/j_E|_v < 1$.



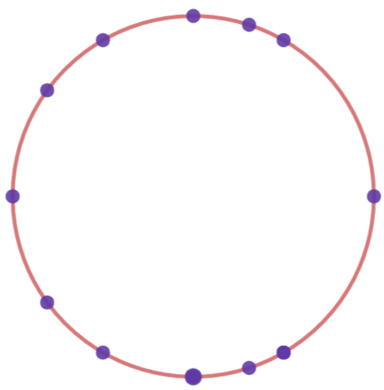
⋆ **If $P, Q \in E(K)$ map to distinct places on the circle, then their positions completely determine $\lambda_v(P - Q)$.**

Outline of Hindry–Silverman

Within this bad reduction situation, two cases:

1. $v(q) = v(1/j_E) \leqslant 6$
2. Otherwise.

Case (1) tells us that points $P \in E(K)$ can only map to a restricted part of the circle:



whereas (2) imposes no restriction on the position of the points.

Outline of Hindry–Silverman

Upshot: If $P_1, \dots, P_N \in E(K)$ are pairwise distinct and any significant "proportion" of bad places falls into Case (1), then replace with $Q_1 = [60]P_1, \dots, Q_N = [60]P_N$

If $N \gg 1$ and there are N' distinct Q_i,

$$\frac{1}{N'(N'-1)} \sum_{v \in M_K^0} \sum_{Q_i \neq Q_j} \lambda_v(Q_i - Q_j) \geqslant C \sum_{v \in M_K^0} \log^+ |j_E|_v$$

for some explicit $C > 0$ independent of E.

Outline of Hindry–Silverman

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for some explicit $C > 0$ independent of E.

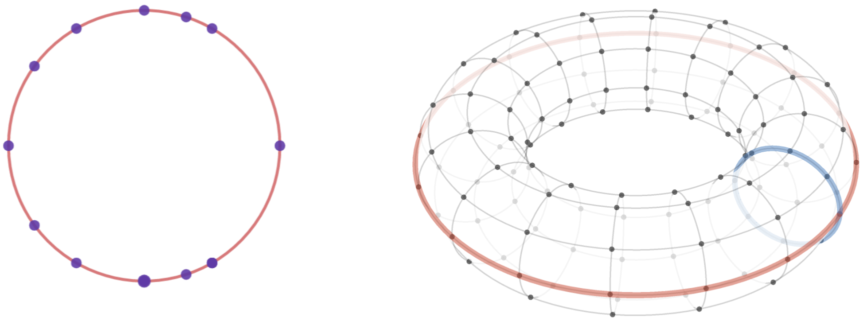
On the other hand, if nearly all bad places fall into Case (2), then Szpiro's Conjecture is violated.

A separate combinatorial argument handles the archimedean places.

Uniform Boundedness in higher dimensions

One might ask whether Hindry–Silverman's approach can be ported to families of higher-dimensional abelian or Jacobian varieties.

- Skeleton:



- Szpiro's conjecture analogue: there are analogous upper bounds on the average number of components of the Néron model at places of bad reduction, which follow from *abc*

Uniform Boundedness in higher dimensions

Problem: normalized local heights don't sum to the global canonical height. Instead,

$$\hat{h}_\Theta(P) = \sum_{v \in M_K} \lambda_{v,\Theta}(P) + \kappa$$

for some κ.

Thus any higher-dimensional analogue of

$$\frac{1}{N'(N'-1)} \sum_{v \in M_K^0} \sum_{Q_i \neq Q_j} \lambda_v(Q_i - Q_j) \geqslant C \sum_{v \in M_K^0} \log^+ |j_E|_v$$

is not useful unless we can also prove this lower bound for $\frac{1}{N'(N'-1)} \sum_{v \in M_K^0} \sum_{Q_i \neq Q_j} \lambda_v(Q_i - Q_j) + \kappa$.

Uniform Boundedness in higher dimensions

Solution: Replace Avg$\lambda_v(P_i - P_j)$ with a generalized Vandermonde matrix evaluated at a certain basis $\{\eta_j\}$ of global sections of $\mathcal{L}^n$ for $\mathcal{L}$ very ample:

$$V_{m,v}(P_1, \ldots, P_m) = -\frac{1}{n} \log \left| \operatorname{Det} \left(\eta_j(\widetilde{P}_i) \right) \right|_v + \sum_i \hat{H}_v(\widetilde{P}_i),$$

where $m = h^0(\mathcal{L}^n)$ and $\hat{H}_v$ is a homogeneous escape-rate function.

Theorem (L., '24)

The functions $V_{m,v}$ satisfy an Elkies-type bound: There exists a C such that for all $n \geqslant 2$, all v and all $P_1, \ldots, P_m$ on the abelian variety,

$$\frac{1}{m} V_{m,v}(P_1, \ldots, P_m) \geqslant \frac{-C \log n}{n}.$$

Uniform Boundedness in higher dimensions

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Remarks:

- This result holds for general polarized dynamical systems.
- A Lehmer-style result on points of small canonical height on abelian varieties follows, over product formula fields having perfect residue fields. The bound has the form

$$\hat{h}_{\mathcal{L}}(P) \geqslant \frac{C'}{[K(P) : K]^{2\dim(A)+3+\epsilon}}.$$

K-rationality in the arithmetic of infinite forward orbits

In the function field setting, the following strengthening of Mordell is well-known.

Theorem (Height Uniformity)

Let X be a nice algebraic curve of genus $\geqslant 2$ over a one-dimensional, characteristic 0 function field K, and let $D \geqslant 1$. There are constants C_1 and C_2 depending on X, K, a chosen height h, and D such that for all $P \in X(L)$ with $[L : K] \leqslant D$,

$$h(P) \leqslant C_1 \cdot \mathrm{genus}(L) + C_2.$$

The constants C_1 and C_2 can be given very explicitly in the case of hyperelliptic curves $y^2 = f(x)$.

K-rationality in the arithmetic of infinite forward orbits

Number field analogue:

Conjecture (Height Uniformity/Discriminant Conjecture)

Let X be an algebraic curve of genus $\geqslant 2$ over a number field K, and let $D \geqslant 1$. There are constants C_1 and C_2 depending on X, K, a chosen height h, and D such that for all $P \in X(L)$ with $[L : K] \leqslant D$,

$$h(P) \leqslant C_1 \cdot \log|\Delta_L| + C_2.$$

The Height Uniformity Conjecture has deep connections to the arithmetic of forward orbits.

A couple of examples:

1. Primitive prime divisors, and hence arboreal representations
2. Liminfs of the Néron–Tate height on curves embedded into their Jacobians (i.e., quantitative Bogomolov)

Primitive prime divisors

We restrict our discussion to polynomial maps $f : \mathbb{P}^1 \to \mathbb{P}^1$.

Definition

We say that a prime $\mathfrak{p}$ *of* K *is a* **primitive prime divisor** *of* $f^n(\alpha)$ *if:*

- $v_{\mathfrak{p}}(f^n(\alpha)) > 0$, *and*
- $v_{\mathfrak{p}}(f^m(\alpha)) \leqslant 0$ *for all* $f^m(\alpha) \neq 0$ *with* $m < n$.

Example: $f(x) = x^2 - 7/4$, $\alpha = 0$

$0 \mapsto -7/4 \mapsto 21/16 \mapsto -7/256 \mapsto -114639/65536$

Here, $f^3(0)$ fails to have a primitive prime divisor.

PPDs: specified multiplicities

Leveling up, we might ask for specific multiplicities in the prime divisors.

Example: the **Sylvester sequence** is given by the forward orbit of 2 under $f(x) = x^2 - x + 1$:

$$2 \mapsto 3 \mapsto 7 \mapsto 43 \mapsto 1807 = 13 \times 139 \cdots$$

It appears that each term in the sequence is squarefree, but this is not yet known to be true.

PPDs: specified multiplicities

A simple trick allows us to connect PPDs to points on higher genus curves.

Suppose

$$f^n(\alpha) = 0 \bmod \mathfrak{p},$$

and that

$$f^k(\alpha) = 0 \bmod \mathfrak{p}$$

for some $0 \leqslant k \leqslant n-1$. As $f^n(\alpha) = f^{n-k}(f^k(\alpha))$, this is saying that

$$f^{n-k}(0) = 0 \bmod \mathfrak{p}.$$

Thus, for any **non**-primitive prime divisor $\mathfrak{p}$ of $f^n(\alpha)$, either

$$\mathfrak{p} \mid f^j(0) \text{ for some } 0 \leqslant j \leqslant \lfloor n/2 \rfloor$$

or

$$\mathfrak{p} \mid f^j(\alpha) \text{ for some } 0 \leqslant j \leqslant \lfloor n/2 \rfloor.$$

PPDs: specified multiplicities

Assume for simplicity that $f^3(X)$ is separable, and that $\mathcal{O}_K$ is a PID.

For α having infinite forward orbit under f, and $n \geqslant 4$, write

$$d_n y_n^2 = f^n(\alpha)$$

with d_n squarefree.

We have

$$d_n y_n^2 = f^3(f^{n-3}(\alpha)),$$

so $(f^{n-3}(\alpha), \sqrt{d_n} y_n)$ is a quadratic point on the higher genus curve

$$Y^2 = f^3(X).$$

The Height Uniformity Conjecture says that for $L = K(\sqrt{d_n})$,

$$\text{genus}(L) \geqslant \frac{1}{C_1}\left(h(f^{n-3}(\alpha)) - C_2\right).$$

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In other words,

$$h(d_n) \gg h(f^{n-3}(\alpha)).$$

Primitive prime divisors: specified multiplicities

OTOH, by our divisibility trick, the product of all of the non-primitive prime divisors is necessarily small:

$$h\left(\prod_{0 \leqslant j \leqslant \lfloor n/2 \rfloor} f^j(\alpha) f^j(0)\right) = O(d^{n/2})$$

whereas $h(f^{n-3}(\alpha)) \approx d^{n-3}\hat{h}_f(\alpha)$.

Provided $f^k(0) \neq 0$ for all k, we thus expect that for all but finitely many n, $f^n(\alpha)$ has a PPD of odd multiplicity.

Remarks:

- Over function fields, this approach works well for *uniform* PPD results.
- For non-uniform results, can use *abc* to show that one has PPDs of multiplicity 1 for all but finitely many n.
- Odd multiplicity PPDs are crucial in large image results for arboreal reps.

Canonical heights on Jacobians and points on higher-genus curves

Another connection to dynamics is seen in effective versions of the Bogomolov conjecture.

Theorem (Zhang,'93)

Let X/K be a nice curve of genus $g \geqslant 2$, and $j : X \hookrightarrow J := \mathrm{Jac}(X)$ an Abel-Jacobi embedding. Let ω_a be the admissible dualizing sheaf on X. Then

$$\liminf_{P \in X(\overline{K})} h_{\mathrm{NT}}(j(P)) \geqslant \frac{\omega_a^2}{4(g-1)}.$$

Hence $\omega_a^2 > 0$ implies the Bogomolov Conjecture.

Remark: ω_a is a more natural analogue of the Arakelov dualizing sheaf.

Canonical heights on Jacobians and points on higher-genus curves

If $K/k(t)$ is a one-dimensional char. 0 function field, then ω_a^2 is known to be commensurate to the total "badness" of the reduction of X.

If δ_v is the v-adic delta-invariant of X for each $v \in M_K$, then there are positive constants C_1, C_2, C_3, C_4 depending only on $g(X)$ and $[K : k(t)]$ such that

$$C_1 \sum_{v \in M_K} n_v \delta_v \leqslant \omega_a^2 \leqslant C_2 \sum_{v \in M_K} n_v \delta_v + C_3 \cdot \mathrm{genus}(K) + C_4. \qquad (\star)$$

Number field case: both inequalities are open! In fact:

Theorem (Moret-Bailly, '90)

The right-hand inequality in ($\star$) implies the Height Uniformity Conjecture.

Canonical heights on Jacobians and points on higher-genus curves

Remark: Moret-Bailly also shows that the Height Uniformity Conjecture implies a weak form of *abc*, namely that (for each K) the *abc* conjecture is true for all sufficiently large ϵ.

Over $\mathbb{Q}$, the *abc* conjecture says:

Conjecture (*abc*)

Let $\epsilon > 0$. There is a C_ϵ such that for any positive coprime integers a, b, c satisfying $a + b = c$,

$$c \leqslant C_\epsilon \left(\prod_{\text{primes } p|abc} p \right)^{1+\epsilon}.$$

Many results conditioned on *abc* in fact only use its truth for all sufficiently large ϵ.

Further questions

1. Other applications of A-G function average lower bound?
2. Unconditional (with or w/o uniformity, with or w/o multiplicities) PPD results over number fields? Aside from examples like:
   - $z^d + c \in \mathbb{Q}[z]$ with $\alpha = 0$ (Krieger, '13)
   - Polys over $\mathbb{Q}$ fixing 0 (Ingram–Silverman, '09)?
3. Upper bound in Conjecture ($\bigstar$) for Galois covers of $\mathbb{P}^1$? Namely

$$\omega_a^2 \leqslant C_2 \sum_{v \in M_K} n_v \delta_v + C_3 \cdot \text{genus}(K) + C_4$$

Thank you!