# Xeric varieties and their relation with geometry and p-adic analysis

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Mordell's conjecture 100 years later
Cambridge, MA, July 12 of 2024

# Faltings' theorem for curves

In this talk $k$ will always be a number field (for instance, $k = \mathbb{Q}$).

**Theorem (Mordell's conjecture 1922, first proved by Faltings 1983)**

*Let $X/k$ be a smooth projective curve of genus $g \geq 2$. Then $X(k)$ is finite*

**Definition.** (Lang) A variety $X/k$ is **Mordellic** if for every $L/k$ the set $X(L)$ is finite.

**Question.** Is there a geometric description of Mordellic varieties?

# Towards a geometric characterization of Mordellic varieties

**Conjecture (Green–Griffiths–Lang)**

*Let $X/k$ be a smooth projective variety. The following are equivalent:*

(i) *$X$ is Mordellic;*

(ii) *Every rational map $A \dashrightarrow X_{\overline{k}}$ over $\overline{k}$ is constant, with $A$ varying over abelian varieties.*

(iii) *$X(\mathbb{C})$ is hyperbolic (for some/all embeddings $k \to \mathbb{C}$): every holomorphic map $f : \mathbb{C} \to X(\mathbb{C})$ is constant.*

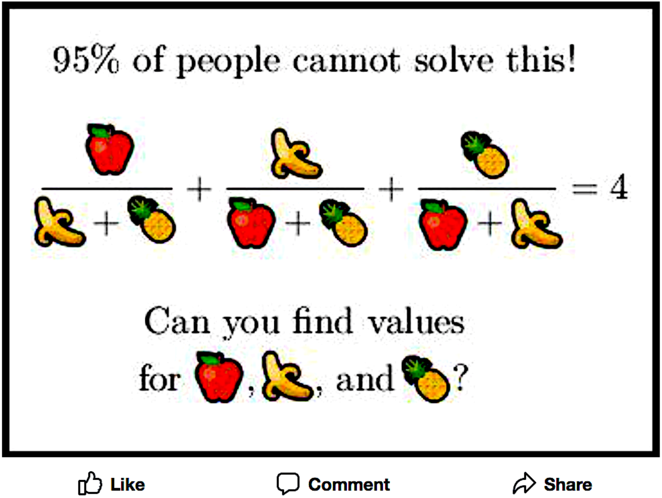
# Towards a geometric characterization of Mordellic varieties

**Theorem (Faltings 1991; Kawamata, Ueno, Green, Bloch, ...)**

*The Green–Griffiths–Lang conjecture holds for subvarieties of abelian varieties.*

All results in the literature towards the Green–Griffiths–Lang conjecture **that include (i) (that is: $X$ Mordellic)** are in some way a consequence of this theorem or of another theorem of Faltings (the **Shafarevich conjecture** for abelian varieties).

# A math challenge from Facebook



# A math challenge from Facebook

Let $X =$ apples, $Y =$ bananas, $Z =$ pineapples.

If you are used to Facebook math-fruit challenges, then you would know that they are positive integers.

The equation becomes

$$\frac{X}{Y+Z} + \frac{Y}{X+Z} + \frac{Z}{X+Y} = 4.$$

Can you see a solution?

# A math challenge from Facebook

The smallest solution is:

$$\begin{aligned} X =& 4373612677928697257861252602371390152816\text{5}\\ & 3755816161361862143799233784234677720\text{36}\\ Y =& 3687513179412999982719781156522547482549297\\ & 9968971970996283137471637224634055579\\ Z =& 1544768021087461664419513150199198374856643\\ & 2566956543170002663489825320203527799\text{9} \end{aligned}$$

The problem defines an elliptic curve of positive rank; there are infinitely many solutions but they are too sparse! (source: Bremner–Macleod 2014)

**Moral of the story:** Finiteness is a key qualitative aspect of Diophantine equations, but sparsity is interesting in its own right and it is not always explained by finiteness.

# Counting rational points

Let $X/k$ be a projective variety and $\mathcal{A}$ an ample line bundle on $X$. Attached to this there is a height function

$$H_{\mathcal{A}}(-) : X(\bar{k}) \to \mathbb{R}.$$

For a Zariski open set $U \subseteq X$ and a number field $L/k$ we define

$$N_{\mathcal{A}}(U, L, T) = \#\{P \in U(L) : H_{\mathcal{A}}(P) \leq T\}$$

which is finite for every $T > 0$ because $\mathcal{A}$ is ample (Northcott property).

# Counting rational points

For instance, if $X$ is a smooth projective curve of genus $g$:

- For $g = 0$ we have $N_{\mathcal{A}}(X, L, T) \sim c_1 \cdot T^{c_2}$ where $c_j = c_j(\mathcal{A}, L) > 0$.
- For $g = 1$ we have $N_{\mathcal{A}}(X, L, T) \sim c_1 \cdot (\log T)^{c_2}$ where $c_j = c_j(\mathcal{A}, L) \geq 0$
- For $g \geq 2$ we have that $N_{\mathcal{A}}(X, L, T)$ is bounded as $T \to \infty$.

# Sparsity and xeric varieties

**Definitions.** Let $X/k$ be a projective variety, $U \subseteq X$ open.

- For $L/k$ number field we say that $U(L)$ is **sparse** if for every $\epsilon > 0$ we have
$$N_{\mathcal{A}}(U, L, T) \ll_\epsilon T^\epsilon.$$

- We say that $U$ is **xeric** if $U(L)$ is sparse for every number field $L/k$.

For instance, rational curves (genus 0) are not xeric, while curves of genus $g \geq 1$ are xeric.
The terminology comes from biology.

# Compare to genus $g = 0$

Selva Valdiviana. South of Chile.

Not a xeric biome: abundant vegetation.

# Compare to genus $g = 1$

Pampa del Tamarugal. Center-north of Chile.
Xeric biome: very dry with sparse vegetation.

# Compare to genus $g \geq 2$

Desierto de Atacama, north of Chile.

Xeric biome: the driest place on earth other than the poles, almost no vegetation at all.

# Towards a geometric characterization of xeric varieties

Silly reason for having a non-xeric variety: **if $X$ contains a rational curve**.

**Conjecture (Implied by a conjecture of Manin)**

*Let $X/k$ be a smooth projective variety. Then $X$ is xeric if and only if it contains no rational curves over $\overline{k}$.*

(The "$\Rightarrow$" direction is clear.)

**Theorem (McKinnon 2010)**

*Vojta's conjectures and the conjectures in the Minimal Model Program imply the previous conjecture.*

So, it is likely to be true.

# Analytic counterpart

The previous conjecture relates arithmetic and geometry. But the Green–Griffiths–Lang conjecture also had a complex analytic component.

What about xeric varieties?

Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$.

**Conjecture (Cherry 1994)**

*Let $X/\mathbb{C}_p$ be a smooth projective variety. Then $X$ contains a rational curve if and only if there is a non-constant p-adic analytic map $f\colon \mathbb{C}_p \to X$.*

(The "$\Rightarrow$" direction is clear.)

# Analytic counterpart

**Conjecture (Cherry 1994)**

*Let $X/\mathbb{C}_p$ be a smooth projective variety. Then $X$ contains a rational curve if and only if there is a non-constant p-adic analytic map $f\colon \mathbb{C}_p \to X$.*

Classical progress on this:

- Berkovich 1990: Curves of genus $g \geq 1$ admit no non-constant $p$-adic analytic map $f\colon \mathbb{C}_p \to X$.
- Cherry 1994: Abelian varieties admit no non-constant $p$-adic analytic map $f\colon \mathbb{C}_p \to X$.

# Final form of the conjecture

## Conjecture (Main conjecture)

*Let $X/k$ be a smooth projective variety. The following are equivalent:*

(1) *$X$ is xeric.*

(2) *$X$ contains no rational curve over $\overline{k}$*

(3∃) *For some prime $p$ and some embedding $k \to \mathbb{C}_p$, every $p$-adic analytic map $f\colon \mathbb{C}_p \to X(\mathbb{C}_p)$ is constant.*

(3∀) *For every prime $p$ and embedding $k \to \mathbb{C}_p$, every $p$-adic analytic map $f\colon \mathbb{C}_p \to X(\mathbb{C}_p)$ is constant.*

# Étale covers

A useful fact that we will use many times:

**Lemma**

*The Main Conjecture is compatible with étale covers.*

In several cases, this allows one to bootstrap arguments.
The previous lemma actually consists of several variations of different compatibilities. Here is one instance:

**Lemma ($p$-adic analytic maps and étale covers)**

*Let $\pi : Y \to X$ be a finite étale map of smooth projective varieties over $\mathbb{C}_p$ and let $f : \mathbb{C}_p \to X$ a $p$-adic analytic map. Then there is a lift of $f$, namely, there is $g : \mathbb{C}_p \to Y$ such that $\pi \circ g = f$.*

# Some special sets

Let $X/k$ be as before. Define:

- $Z_{rc}(X)$ the Zariski closure of all rational curves over $\overline{k}$ in $X$.
- $Z_{p-an}(X)$ the Zariski closure of the images of all non-constant $p$-adic analytic maps $f\colon \mathbb{C}_p \to X(\mathbb{C}_p)$.

Clearly $Z_{rc} \subseteq Z_{p-an}$.

**Question**

*Do we always have $Z_{rc} = Z_{p-an}$ ? Or at least $Z_{rc} = \cap_p Z_{p-an}$ ?*

# A variation: describing the largest xeric open set

**Conjecture**

*Let $X/k$ be a smooth projective variety. Then $X - Z_{rc}$ is xeric (and it is the largest xeric open set in $X$).*

McKinnon's work also shows that this would follow from Vojta's conjecture and the MMP.

# Some known cases

Using the Mordell–Weil theorem and Cherry's theorem we get:

**Theorem**

*The main conjecture is true for abelian varieties. They satisfy all the listed properties.*

A less obvious case:

**Theorem (GF-P)**

*The main conjecture is true for varieties with nef tangent bundle.*

**Idea.** By Demailly–Peternell–Schneider 1994, up to étale cover such varieties are Fano fibrations over abelian varieties.

- If the fibres are positive dimensional apply Mori's theorem to get plenty of rational curves.
- Otherwise we have an abelian variety. Use the previous result. □

# Weak positivity of the cotangent bundle

When $\Omega^1_X$ is nef, $X$ contains no rational curves. So the Main Conjecture suggests:

**Conjecture (The case of nef cotangent bundle)**

*Let $X/k$ be a smooth projective variety with $\Omega^1_X$ nef. Then:*

(a) *$X$ is xeric.*

(b) *$Z_{p-an} = \emptyset$ for every $p$.*

Adapting methods of McKinnon we prove that (a) follows from Vojta's conjecture (MMP not required here.)

# Weak positivity of the cotangent bundle

Let us discuss (b).
Recall:

**Theorem (Kobayashi)**

*If $X$ is a complex projective manifold with ample $\Omega^1_X$, then it is hyperbolic: every holomorphic $f\colon \mathbb{C} \to X$ is constant.*

In the $p$-adic setting we use Nevanlinna theory to prove

**Theorem (GF-P)**

*If $X$ is a p-adic projective manifold with semi-ample $\Omega^1_X$, then $Z_{p-an} = \emptyset$: every p-adic analytic map $f\colon \mathbb{C}_p \to X$ is constant.*

Regarding these positivity conditions we recall:

$$\text{ample} \Rightarrow \text{semi-ample} \Rightarrow \text{nef}.$$

# Large fundamental groups

For a variety $X$ we let $\widehat{\pi}_1(X)$ be the algebraic fundamental group; it classifies étale covers of $X$.

**Definition.** (Kollar 1993) Let $X$ be a smooth projective variety over $\mathbb{C}$. We say that $\widehat{\pi}_1(X)$ is **large** if for every positive dimensional sub-variety $Y \subseteq X$ the map $\widehat{\pi}_1(Y') \to \widehat{\pi}_1(X)$ has infinite image, where $Y'$ is a desingularization of $Y$.

**Examples.**

- If $X$ is a curve, it has large fundamental group if and only if $g_X \geq 1$.
- Abelian varieties
- Compact ball quotients (e.g. fake projective planes)

# Large fundamental groups

After the work of Ellenberg–Lawrence–Venkatesh 2021, we have the following result by Brunebarbe–Maculan

**Theorem (Brunebarbe–Maculan 2022)**

*Let $X/k$ be a smooth projective variety with large fundamental group over $\mathbb{C}$. Then $X$ is xeric.*

**Idea.**

- Using largeness, construct étale covers of $X$ with the property that **all** its positive dimensional subvarieties have large degree.
- Projective varieties of large degree tend to have slowly growing counting functions for their rational points.

# Large fundamental groups

**Corollary**

*If $X/k$ is a compact ball quotient, then the Main Conjecture holds for $X$.*

**Proof.** They are xeric by the previous result.
They have ample $\Omega^1_X$, in particular semi-ample, so $Z_{p-an} = \emptyset$ for all $p$. □

# Infinite fundamental group

What if we don't assume largeness of fundamental groups?

**Theorem (GF-P)**

*Let $X/k$ be a smooth projective surface of general type with* ***infinite*** $\widehat{\pi}_1(X)$*. Depending on the case we make one of the following assumptions:*

- *If $\widehat{q}(X) = 0$, we require that the topological universal cover of $X(\mathbb{C})$ is holomorphically convex (Shafarevich conjectured that this is always the case);*
- *If $\widehat{q}(X) = 1$, we assume the classical conjecture that the 2-torsion part of class groups of number fields is small (in a technical sense);*
- *If $\widehat{q}(X) \geq 2$ we don't need additional assumptions.*

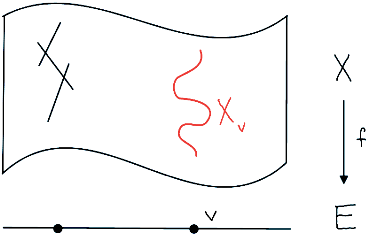
*Then $X$ has a non-empty xeric Zariski open set.*

# The case $\widehat{q}(X) = 1$: fibrations over elliptic curves

The most complicated case is $\widehat{q}(X) = 1$.

Up to an étale cover we can assume that the Albanese map is $f\colon X \to E$ with $E$ an elliptic curve and general fibre a curve of genus $g \geq 2$.

We count points $P \in X(k)$ with $H_{\mathcal{A}}(P) \leq T$ fibre by fibre:




- By Dimitrov–Gao–Habegger 2020:

$$\#X_v(k) < c^{1+\mathrm{rk}\, Jac(X_v)}$$

- Add over $v \in E(k)$ of bounded height.
- The assumption on class groups helps to control $\mathrm{rk}\, Jac(X_v)$. □

# The $s$-invariant of the diagonal

Let $\mathcal{A}$ be an ample line bundle on $X$. Let $\Delta \subseteq X \times X$ be the diagonal and $b : Y \to X \times X$ the blow-up along $\Delta$ with exceptional divisor $E$.

As a measure of positivity of $\Delta$ one defines

$$s(X, \mathcal{A}) := \inf\{t \geq 0 : t \cdot b^*(\mathcal{A} \boxtimes \mathcal{A}) - E \text{ is nef}\}.$$

**Remarks.**

- We always have $s(X, \mathcal{A}) > 0$.
- $s(X, \mathcal{A})$ is the reciprocal of the Seshadri constant of $\Delta$ with respect to $\mathcal{A}$.

# The $s$-invariant of the diagonal

Inspired by the work of McKinnon, Tanimoto proved:

**Theorem (Tanimoto 2019)**

*Let $X/k$ be a smooth projective variety of dimension $n$ and $\mathcal{A}$ an ample line bundle on $X$. For every $\epsilon > 0$ we have*

$$N_{\mathcal{A}}(X, k, T) \ll T^{2n \cdot s(X,\mathcal{A}) + \epsilon}.$$

The measure of positivity of the diagonal is used to show repulsion between rational points, hence, they are well-spaced.

# The $\widehat{s}$-invariant of the diagonal

Let us introduce a variation of the $s$-invariant:

$$\widehat{s}(X, \mathcal{A}) = \inf\{s(X', p^*\mathcal{A}) : p : X' \to X \text{ is étale}\}$$

## Theorem (GF-P)

*Let $X/k$ be a smooth projective variety of dimension $n$ and $\mathcal{A}$ an ample line bundle on $X$. For every $\epsilon > 0$ we have*

$$N_{\mathcal{A}}(X, k, T) \ll T^{2n \cdot \widehat{s}(X, \mathcal{A}) + \epsilon}.$$

## Corollary

*If $\widehat{s}(X, \mathcal{A}) = 0$ then $X$ is xeric.*

# The $\widehat{s}$-invariant of the diagonal

The previous theorem might seem like a small variation of Tanimoto's result, but there is a crucial difference:

$$s(X, \mathcal{A}) > 0 \text{ always, while } \widehat{s}(X, \mathcal{A}) \textbf{ can vanish}.$$

**Examples where $\widehat{s}(X, \mathcal{A})$ vanishes:**

- Abelian varieties
- Compact ball quotients (for instance, fake projective planes)
- Conjecturally, if the fundamental group is large (Hwang).

This gives a new proof of the Main Conjecture for compact ball quotients.

# Some questions

As we explained, we have many more methods for showing that a variety is xeric: large fundamental groups, the Albanese map, the $\widehat{s}$-invariant. But in all these cases we need (at least) infinite $\widehat{\pi}_1(X)$.

**Question**

*Consider $X/k$ smooth projective of positive dimension. Can one show that $X$ is xeric in some case with $\widehat{\pi}_1(X)$ finite?*

# Some questions

Cherry conjectured that no $p$-adic analytic map to a K3 surface has Zariski dense image. We propose the following generalization:

**Conjecture**

*Let $X/\mathbb{C}_p$ be smooth projective such that there is a p-adic analytic map $f\colon \mathbb{C}_p \to X$ with Zariski dense image. Then $\kappa(X) = -\infty$.*

After presenting this talk at Journées arithmétiques 2023, Daniel Litt asked the following:

**Question (D. Litt)**

*Let $X/\mathbb{C}_p$ be smooth projective such that there is a p-adic analytic map $f\colon \mathbb{C}_p \to X$ with Zariski dense image. Is $X$ unirational? or at least uniruled?*

Thanks for your attention.