Density of rational points near manifolds

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The Mordell conjecture 100 years later
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Counting rational points on projective varieties

Let $n \geq 2$ and $V \subset \mathbb{P}^n_{\mathbb{Q}}$ an irreducible variety of degree d.
For a point $x = (x_0 : \ldots : x_n) \in \mathbb{P}^n(\mathbb{Q})$ with $x_0, \ldots, x_n \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$ set

$$H(x) = \max_{0 \leq i \leq n} |x_i|.$$

Question

Define for $B \in \mathbb{R}_{\geq 0}$ the counting function

$$N_V(B) := \sharp\{x \in V(\mathbb{Q}) : H(x) \leq B\}.$$

What can we say about upper bounds for $N_V(B)$?

Trivial upper bound: $N_V(B) \ll_V B^{\dim V + 1}$.

Points of bounded height on projective space

For a point $x = (x_0 : \ldots : x_n) \in \mathbb{P}^n(\mathbb{Q})$ with $x_0, \ldots x_n \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$ set

$$H(x) = \max_{0 \leq i \leq n} |x_i|.$$

With this we find that

$$N_{\mathbb{P}^n}(B) = \sharp\{x \in \mathbb{P}^n(\mathbb{Q}) : H(x) \leq B\}$$
$$= \frac{1}{2}\sharp\{(x_0, \ldots, x_n) \in \mathbb{Z}^n, |x_i| \leq B, 0 \leq i \leq n, \ \gcd(x_0, \ldots, x_n) = 1\}$$

Theorem

For $n \geq 1$ we have

$$N_{\mathbb{P}^n}(B) \sim \frac{2^n}{\zeta(n+1)} B^{n+1}.$$

Dimension growth conjecture

Example

Rational linear subspaces, e.g. $V \subset \mathbb{P}^n_{\mathbb{Q}}$ given by

$$a_0 x_0 + \ldots + a_n x_n = 0,$$

with $a_i \in \mathbb{Z}$, $1 \leq i \leq n$ and $a_0 \neq 0$. Then

$$N_V(B) \sim B^n.$$

Example

Let $a_0, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ and $V \subset \mathbb{P}^n_{\mathbb{Q}}$ given by

$$a_0 x_0^2 + \ldots + a_n x_n^2 = 0.$$

Then, for $n \geq 4$ we have

$$N_V(B) \sim cB^{n-1}.$$

Dimension growth conjecture

Hypersurfaces of larger degree

Example

Let $a_0, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ and $V \subset \mathbb{P}^n_{\mathbb{Q}}$ given by

$$a_0 x_0^d + \ldots + a_n x_n^d = 0.$$

By our naive heuristic, we expect

$$N_V(B) \sim cB^{n+1-d}.$$

By the resolution of Vinogradov's mean value theorem due to Bourgain, Demeter and Guth and Wooley, one can prove this asymptotic for $n \gg d^2$.

Dimension growth conjecture

Hypersurfaces with singularities

Example

Let $F_0(\mathbf{x}), F_1(\mathbf{x}) \in \mathbb{Z}[x_0, \ldots, x_n]$ and $V \subset \mathbb{P}^n_{\mathbb{Q}}$, $n \geq 2$ given by

$$x_0 F_0(\mathbf{x}) - x_1 F_1(\mathbf{x}) = 0.$$

Then

$$\begin{aligned} N_V(B) &\geq \frac{1}{2}\sharp\{(x_2, \ldots, x_n) \in \mathbb{Z}^{n-1} : \max_{1 \leq i \leq n} |x_i| \leq B \\ &\quad \gcd(x_2, \ldots, x_n) = 1\} \\ &\gg B^{n-1} \end{aligned}$$

Dimension growth conjecture

Conjecture (Weak Dimension growth conjecture)

Let $V \subset \mathbb{P}^n_{\mathbb{Q}}$ be an irreducible projective variety of degree $\deg(V) \geq 2$. *Then*

$$N_V(B) \ll_V B^{\dim V + \varepsilon},$$

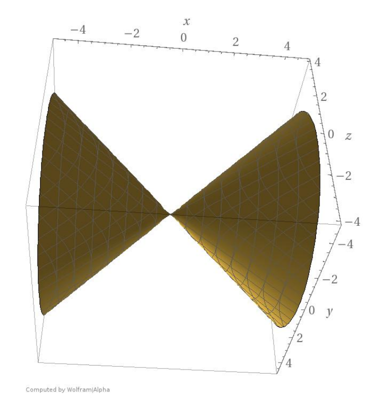
for any $\varepsilon > 0$.

Remark

- Sharper version with uniformity in V and implicit constants only depending on n, d, ε.
- Solved in a series of articles by Browning, Heath-Brown and Salberger, using the determinant method.

Rational points close to manifolds

Example



The surface $\mathcal{S}$ given by the quadratic form $Q(x, y, z) = x^2 - y^2 - \sqrt{2}z^2$.

Rational points close to manifolds

Let $\mathcal{M} \subset \mathbb{R}^n$ be a bounded submanifold with $\dim \mathcal{M} = m$.

A counting function

Let $Q > 1$ and $0 \leq \delta < \frac{1}{2}$. We define the counting function

$$N_{\mathcal{M}}(Q, \delta) := \sharp \left\{ \frac{\mathbf{p}}{q} \in \mathbb{Q}^n : 1 \leq q \leq Q : \operatorname{dist}\left(\frac{\mathbf{p}}{q}, \mathcal{M} \right) \leq \frac{\delta}{q} \right\},$$

where $\mathbf{p} \in \mathbb{Z}^n$.

Trivial upper bound

$$N_{\mathcal{M}}(Q, \delta) \ll Q^{m+1}.$$

Rational points close to manifolds

Example

For $\mathcal{M}$ a piece of a rational linear subspace we have $N_{\mathcal{M}}(Q,\delta) \gg Q^{m+1}$.

Question

Typical expectation for $N_{\mathcal{M}}(Q,\delta)$?

Set $R = n - m$. A heuristic argument via volume computation:

$$N_{\mathcal{M}}(Q,\delta) \sim \left(\frac{\delta}{Q}\right)^R QQ^n \sim \delta^R Q^{m+1}.$$

Rational points close to manifolds

Heuristic

$$N_{\mathcal{M}}(Q,\delta) \sim \delta^R Q^{m+1}.$$

For what size of δ is this realistic?

Example

a) Let $\mathcal{P} \subset \mathbb{R}^2$ be given by $y = x^2$ for $0 \leq x \leq 1$. Then

$$N_{\mathcal{P}}(Q,\delta) \geq N_{\mathcal{P}}(Q,0) \geq \sum_{q \leq \sqrt{Q}} q \gg Q.$$

This suggests $\delta \gg Q^{-1}$ for the 'volume term' to dominate.
b) Let $\mathcal{S} = S^{n-1} \subset \mathbb{R}^n$, $n \geq 3$ be given by $x_1^2 + \ldots + x_n^2 = 1$. Then

$$N_{\mathcal{S}}(Q,0) \sim c_n Q^{n-1}.$$

Rational points close to manifolds

Conjecture

If $\mathcal{M}$ is a bounded submanifold of $\mathbb{R}^n$ with boundary and proper curvature conditions, then for $\varepsilon > 0$

$$N_{\mathcal{M}}(Q,\delta) \ll \delta^R Q^{m+1} + Q^{m+\varepsilon},$$

for all $0 \leq \delta < 1/2$ and $Q \geq 1$. Moreover, if $\delta \geq Q^{-\frac{1}{R}+\varepsilon}$ for some $\varepsilon > 0$, then

$$N_{\mathcal{M}}(Q,\delta) \sim c\delta^R Q^{m+1},$$

for some constant c depending on $\mathcal{M}$, where $Q \to \infty$.

Example (The curvature condition is necessary)

For the Fermat curve $\mathcal{F}_d : \ x^d + y^d = 1$ one has

$$N_{\mathcal{F}_d}(Q,\delta) \gg \delta^{\frac{1}{d}} Q^{2-\frac{1}{d}}.$$

Rational points close to manifolds

Theorem (J.-J. Huang 2019)

The main conjecture holds for smooth compact hypersurfaces $\mathcal{S} \subset \mathbb{R}^n$ with Gaussian curvature bounded away from 0.

Previous work: Beresnevich, Dickinson, Velani, Huxley, Vaughan, Velani, Zorn...

Corollary (J.-J. Huang 2019)

Analogue of dimension growth conjecture for smooth hypersurfaces with non-vanishing Gaussian curvature.

Question

What happens in higher codimension? Can we relax the curvature condition?

Formulating a counting problem

Let $f_1, \dots, f_R$ be smooth functions on $\mathbb{R}^m$ and let $\mathcal{M} \subset \mathbb{R}^{m+R}$ be given by

$$\mathcal{M} = \{(\mathbf{x}, f_1(\mathbf{x}), \dots, f_R(\mathbf{x})) \in \mathbb{R}^{m+R},\ \mathbf{x} \in B_\varepsilon(\mathbf{x}_0)\},$$

where $B_\varepsilon(\mathbf{x}_0)$ is a ball of radius ε around $\mathbf{x}_0 \in \mathbb{R}^m$.

Question

If we want to count rational points close to $\mathcal{M}$, how can we formulate this as a counting problem?

To make things analytically better behaved, we first introduce a smooth weight.

Formulating a counting problem

Notation: for $\alpha \in \mathbb{R}$ write $\|\alpha\| = \min_{n\in\mathbb{Z}} |\alpha - n|$.
Let $m \geq 2$ and let $w \in C_0^\infty(\mathbb{R}^m)$ be a non-negative weight function. For $Q \in \mathbb{N}$ and $\delta \geq 0$, we define

$$\mathcal{N}_w(Q,\delta) = \sum_{\substack{\mathbf{a}\in\mathbb{Z}^m \\ q\leq Q \\ \|qf_1(\mathbf{a}/q)\|\leq\delta \\ \vdots \\ \|qf_R(\mathbf{a}/q)\|\leq\delta}} w\left(\frac{\mathbf{a}}{q}\right).$$

Remark

Note that we expect $\sum_{\mathbf{a}\in\mathbb{Z}^m, q\leq Q} w\left(\frac{\mathbf{a}}{q}\right) \sim Q^{m+1}$, and

$$\mathcal{N}_w(Q,\delta) \sim \delta^R Q^{m+1}.$$

An analogue for the curvature condition

In the case $R = 1$, the manifold parametrised by

$$\mathcal{M} = \{(\mathbf{x}, f_1(\mathbf{x})) \in \mathbb{R}^{m+1},\ \mathbf{x} \in B_\varepsilon(\mathbf{x}_0)\},$$

has non-zero Gaussian curvature in $\mathbf{x}_0$ iff

$$\det H_{f_1}(\mathbf{x}_0) \neq 0,$$

where H_{f_1} is the Hessian matrix of f_1.

Assumption *

Given any $(t_1, \ldots, t_R) \in \mathbb{R}^R \backslash \{\mathbf{0}\}$,

$$\det H_{t_1 f_1 + \cdots + t_R f_R}(\mathbf{x}_0) \neq 0$$

Formulating a counting problem in higher codimension

Theorem (S.-Yamagishi 2022)

Let $m \geq 2$. Assume that Assumption holds and that the support of w is sufficiently small. Then for any $\varepsilon > 0$*

$$\left| \mathcal{N}_w(Q,\delta) - \sigma \delta^R Q^{m+1} \right| \ll Q^{m - \frac{(m-2)(R-1)}{m+2(R-1)} + \varepsilon},$$

where $\sigma > 0$.

Remark

Compare this to the analogue of the dimension growth conjecture which would predict in this situation that

$$N_w(Q,0) \ll Q^{m+\varepsilon}.$$

What if the curvature vanishes?

Consider a hypersurface of the form

$$\mathcal{M} = \{(\mathbf{x}, f(\mathbf{x})),\ \mathbf{x} \in B_1(\mathbf{0})\},$$

where $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is assumed to be a smooth function outside of $\mathbf{0}$, homogeneous of degree D, i.e.

$$f(\lambda \mathbf{x}) = \lambda^D f(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1},\ \lambda \in \mathbb{R}_{>0}.$$

Theorem (Srivastava, N. Technau 2023)

Assume that $n \geq 3$, $\frac{2n-2}{2n-3} < D < n-1$ *and* $H_f(\mathbf{x}) \neq 0$, *for* $\mathbf{x} \neq \mathbf{0}$. *Then for* $\delta \gg Q^{\varepsilon-1}$, *one has*

$$N_w(Q, \delta) \asymp \delta Q^n + \left(\frac{\delta}{Q}\right)^{\frac{n-1}{D}} Q^n.$$

Example: take $f(\mathbf{x}) = \|\mathbf{x}\|_2^D$.

What if the curvature vanishes?

Let $\mathcal{M} \subset \mathbb{R}^n$, where $n \geq 2$, be a bounded submanifold of the form

$$\mathcal{M} = \{(\mathbf{x}, \mathbf{F}(\mathbf{x})), \mathbf{x} \in B_1(\mathbf{0})\},$$

where $B_1(\mathbf{0})$ is the m-dimensional unit ball, and $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), \ldots, F_R(\mathbf{x}))$.

Definition

We say that $\mathcal{M}$ as above is l-nondegenerate at a point $\mathbf{x}_0 \in B_1(\mathbf{0})$, if the partial derivatives of $(\mathbf{x}, \mathbf{F}(\mathbf{x}))$ of order up to l in the point $\mathbf{x}_0$ span $\mathbb{R}^n$.
Moreover, we say that the manifold is l-nondegenerate, if it is l-nondegenerate at almost every point $\mathbf{x}_0 \in B_1(\mathbf{0})$.

l-nondegenerate manifolds

Let $\mathcal{M} \subset \mathbb{R}^n$, where $n \geq 2$, be a bounded submanifold of the form

$$\mathcal{M} = \{(\mathbf{x}, \mathbf{F}(\mathbf{x})), \mathbf{x} \in B_1(\mathbf{0})\},$$

and let Ω, ω, w be suitable weight functions.
Define the counting function

$$N_{\mathbf{F}}(\delta, Q) := \sum_{(q,\mathbf{a}) \in \mathbb{Z}^{m+1}} \Omega\left(\frac{\mathbf{a}}{q}\right) \omega\left(\frac{q}{Q}\right) \prod_{1 \leq i \leq R} w\left(\frac{\|qF_i(\frac{\mathbf{a}}{q})\|}{\delta}\right),$$

where $\|y\| = \min_{k \in \mathbb{Z}} |y - k|$ for $y \in \mathbb{R}$.

l-nondegenerate manifolds

$\mathcal{M} \subset \mathbb{R}^n$, $m = \dim \mathcal{M}$, $R = n - m$.

Theorem (Beresnevich 2012)

Suppose $\mathcal{M} = \{(\mathbf{x}, \mathbf{F}(\mathbf{x})), \mathbf{x} \in B_1(\mathbf{0})\}$ *is l-nondegenerate and analytic. For* $\delta \gg Q^{-\frac{1}{R}}$ *one has*

$$N_{\mathbf{F}}(\delta, Q) \gg \delta^R Q^{m+1}.$$

Beresnevich and Yang 2023:
Suppose $\mathcal{M}$ is l-nondegenerate and smooth. Then outside of a 'small' part of $\mathcal{M}$ one has the upper bound

$$\widetilde{N}_{\mathbf{F}}(\delta, Q) \ll \delta^R Q^{m+1}.$$

- Using a quantitative non-divergence estimate by Bernik, Kleinbock and Margulis.

l-nondegenerate manifolds

$\mathcal{M} \subset \mathbb{R}^n$, $m = \dim \mathcal{M}$, $R = n - m$.

Theorem (S., Srivastava and Technau 2023)

Suppose $\mathcal{M} = \{(\mathbf{x}, \mathbf{F}(\mathbf{x})), \mathbf{x} \in B_1(\mathbf{0})\}$ *is l-nondegenerate and smooth. Then there exists a constant c and a constant $\eta > 0$ such that for*

$$\delta \gg Q^{-\frac{1}{2Rml(n+1)}},$$

we have

$$N_{\mathbf{F}}(\delta, Q) = c\delta^R Q^{m+1} + O\left(\delta^R Q^{m+1-\eta}\right).$$

Remark

We obtain lower bounds of the expected order of magnitude for $\delta \gg Q^{-\frac{3}{2n-1}+\epsilon}$.

l-nondegenerate manifolds

Theorem (S., Srivastava and Technau 2023)

Suppose $\mathcal{M} = \{(\mathbf{x}, \mathbf{F}(\mathbf{x})), \mathbf{x} \in B_1(\mathbf{0})\}$ *is l-nondegenerate and smooth. Then for* $0 \leq \delta \leq 1/2$ *we have*

$$N_{\mathbf{F}}(Q, \delta) \ll \delta^R Q^{m+1} + Q^{m+1-\frac{2R}{2mR(2l-1)(n+1)+2n-1}},$$

and in particular

$$N_{\mathbf{F}}(Q, 0) \ll Q^{m+1-\frac{1}{2lR(n+1)}}.$$

Application: upper bounds for the number of rational points of bounded height on projective varieties, if the corresponding manifold is l-nondegenerate.

Our approach via Fourier analysis

By Poisson summation we find that

$$N_{\mathbf{F}}(Q,\delta) - c\delta^R Q^{m+1} \approx \delta^R Q^{m+1} \sum_{\substack{(\mathbf{v},c)\in\mathbb{Z}^{n+1} \\ \|\mathbf{v}\|_\infty, |c| \sim \delta^{-1}}} I(\mathbf{v},c)$$

with

$$I(\mathbf{v},c) = \int_{\mathbb{R}^{m+1}} y^R e^{2\pi i y Q[\langle(\mathbf{x},\mathbf{F}(\mathbf{x})),\mathbf{v}\rangle - c]} \Omega(\mathbf{x})\omega(y) d\mathbf{x} dy.$$

Proof ingredients:

- Construction of a weight function which separates 'generic parts' and 'special parts'
- For the special parts: use a quantitative non-divergence estimate due to Bernik, Kleinbock and Margulis.
- for the generic parts: rapid decay estimates

Diophantine approximation

Theorem (Dirichlet)

Let $\theta \in \mathbb{R}$ and $Q \in \mathbb{N}$. Then there are integers a, q with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$ such that

$$\left|\theta - \frac{a}{q}\right| < \frac{1}{qQ}.$$

Corollary

Let $\theta \in \mathbb{R}$. Then there are infinitely many tuples $(a, q) \in \mathbb{Z} \times \mathbb{N}$ such that

$$\left|\theta - \frac{a}{q}\right| < \frac{1}{q^2}.$$

Multi-dimensional version: Let $\theta_1, \ldots, \theta_n \in \mathbb{R}$. Then there are infinitely many tuples $(\mathbf{a}, q) \in \mathbb{Z}^n \times \mathbb{N}$ such that

$$\left|\theta_i - \frac{a_i}{q}\right| < q^{-1-\frac{1}{n}}, \quad 1 \leq i \leq n.$$

Applications to Diophantine approximation

Definition

Given a function $\psi : (0, +\infty) \to (0, 1)$, we say that a point $\mathbf{y} \in \mathbb{R}^n$ is ψ-*approximable* if the condition

$$\left\| \mathbf{y} - \frac{\mathbf{a}}{q} \right\|_\infty < \frac{\psi(q)}{q} \tag{0.1}$$

holds for infinitely many $(\mathbf{a}, q) \in \mathbb{Z}^n \times \mathbb{N}$.
Write $\mathcal{S}_n(\psi)$ for the set of ψ-approximable points in $\mathbb{R}^n$.

Example

By Dirichlet's theorem we have $\mathcal{S}_n(q^{-\frac{1}{n}}) = \mathbb{R}^n$.

Applications to Diophantine approximation

Notation: We write μ_n for the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

Theorem (Khintchine's theorem)

Let $\psi : (0, +\infty) \to (0, 1)$ be a monotonic approximation function.

- *If $\sum_{q=1}^{\infty} \psi(q)^n < \infty$, then*

$$\mu_n(\mathcal{S}_n(\psi)) = 0.$$

- *If $\sum_{q=1}^{\infty} \psi(q)^n = \infty$, then the complement of $\mathcal{S}_n(\psi)$ has Lebesgue measure zero.*

Example

We recover that the complement of $\mathcal{S}_n(q^{-\frac{1}{n}})$ in $\mathbb{R}^n$ has Lebesgue measure zero.

Khintchine's theorem

Convergence case: Assume that

$$\sum_{q=1}^{\infty} \psi(q)^n < \infty$$

and consider $\mathcal{S}_n(\psi) \cap [0,1]^n$. For $q \in \mathbb{N}$ consider the set

$$E_q(\psi) = \bigcup_{0 \leq \mathbf{a} \leq q} \prod_{i=1}^{n} \left(\frac{a_i}{q} - \frac{\psi(q)}{q}, \frac{a_i}{q} + \frac{\psi(q)}{q} \right).$$

Note that

$$\mu_n(E_q(\psi)) \ll \psi(q)^n.$$

Now apply the convergence case of the Borel-Cantelli lemma.

Applications to Diophantine approximation

Question (Sprindzuk, Kleinbock, Lindenstrauss, Margulis, Weiss)

Let $\mathcal{M} \subset \mathbb{R}^n$ be a nondegenerate submanifold and $\psi : (0, +\infty) \to (0, 1)$ a monotonic approximation function. Assume that

$$\sum_{q=1}^{\infty} \psi(q)^n$$

converges/diverges. Can one show that almost no/almost all points of $\mathcal{M}$ are ψ-approximable?

Theorem (Beresnevich, Yang 2023)

If $\sum_{q=1}^{\infty} \psi(q)^n < \infty$, then almost all points on $\mathcal{M}$ are not ψ-approximable.

Generalisations to Hausdorff measure

For a Lebesgue measurable set $A \subset \mathbb{R}^n$ we write $\dim(A)$ and $\mathcal{H}^s(A)$ for the Hausdorff dimension and the s-dimensional Hausdorff measure.

Theorem (The Jarník-Besicovitch theorem)

Let $\tau \geqslant 1/n$. Then $\dim \mathcal{S}_n(q^{-\tau}) = \frac{n+1}{\tau+1}$.

Theorem (Jarník's theorem)

Given any monotonic function ψ and $0 < s < n$, we have

$$\mathcal{H}^s(\mathcal{S}_n(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^n \left(\frac{\psi(q)}{q}\right)^s < \infty, \\ \infty & \text{if } \sum_{q=1}^{\infty} q^n \left(\frac{\psi(q)}{q}\right)^s = \infty. \end{cases}$$

Generalisations to Hausdorff measure

Question

Find the Hausdorff dimension s of the set $\mathcal{S}_n(\psi) \cap \mathcal{M}$ *for a smooth submanifold* $\mathcal{M} \subset \mathbb{R}^n$.

Question (Dimension Problem)

Let $1 \leqslant m < n$ *be integers and* $\widetilde{\mathcal{M}}_{n,m}$ *be the class of submanifolds* $\mathcal{M} \subset \mathbb{R}^n$ *of dimension m which are nondegenerate at every point. Find the maximal value* $\tau_{n,m}$ *such that*

$$\dim \mathcal{S}_n(q^{-\tau}) \cap \mathcal{M} = \frac{n+1}{\tau+1} - \text{codim } \mathcal{M} \qquad \text{whenever } 1/n \leqslant \tau < \tau_{n,m}$$

for every manifold $\mathcal{M} \in \widetilde{\mathcal{M}}_{n,m}$.

Generalisations to Hausdorff measure

Question (Dimension Problem)

Let $1 \leqslant m < n$ be integers and $\widetilde{\mathcal{M}}_{n,m}$ be the class of submanifolds $\mathcal{M} \subset \mathbb{R}^n$ of dimension m which are nondegenerate at every point. Find the maximal value $\tau_{n,m}$ such that

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for every manifold $\mathcal{M} \in \widetilde{\mathcal{M}}_{n,m}$.

Conjecture (Beresnevich, Yang 2023)

Let $1 < m < n$. Then $\tau_{n,m} = \frac{1}{n-m}$.

Generalisations to Hausdorff measure - new results

Corollary (S., Srivastava and Technau 2023)

Let $n \geq 2$ be an integer, $\tau \in [\frac{1}{n}, 1)$ be a real number, and $\mathcal{M}$ be a smooth l-nondegenerate submanifold of $\mathbb{R}^n$ of dimension m. Suppose that τ satisfies

$$\tau < \frac{3\alpha + 1}{(2n-1)\alpha + n},$$

where $\alpha := \frac{1}{R(2l-1)(n+1)}$. *Then*

$$\dim(\mathcal{M} \cap \mathcal{S}_n(q^{-\tau})) = \frac{n+1}{\tau+1} - \operatorname{codim} \mathcal{M}.$$

Remark: Beresnevich and Yang 2023 obtain the same result under the condition

$$\frac{n\tau - 1}{\tau + 1} \leq \frac{\alpha(3 - 2n\tau)}{2\tau + 1}.$$

Questions

Thank you for listening!