UNIVERSITÄT BAYREUTH

Formalizing Mordell ?

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The Mordell conjecture $2 \cdot 3 \cdot 17$ years later

MIT

July 12, 2024

Prologue: a Challenge

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Challenge′:

Beat $C = 8$ for $g \to \infty$!

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Lean has a large cohesive and actively developed library Mathlib
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comprising most ungergraduate and quite some higher-level mathematics.

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- Avoid mistakes in one's research

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Corollary 9.10. *Suppose that C/k is a smooth projective curve of genus 2 given by an integral Weierstrass model $\mathcal{C}$ such that there are three nodes in the special fiber of $\mathcal{C}$. We say that $\mathcal{C}$ is split if the two components A and E of the special fiber of $\mathcal{C}^{\min}$ are defined over $\mathfrak{k}$; otherwise $\mathcal{C}$ is nonsplit. Let $v(\Delta) = m_1 + m_2 + m_3$ as above and set $M = m_1m_2 + m_1m_3 + m_2m_3$.*

⋮

(c) *If two of the nodes lie in a quadratic extension of $\mathfrak{k}$ and are conjugate over $\mathfrak{k}$ and one is $\mathfrak{k}$-rational, then*

$$\beta = \begin{cases} \frac{m_1}{M} \max\left\{ \left\lfloor \frac{m_1^2}{2} \right\rfloor + m_1m_3, \left\lfloor \frac{m_3^2}{2} \right\rfloor + m_1 \left\lfloor \frac{m_3}{2} \right\rfloor \right\} & \textit{if } \mathcal{C} \textit{ is split,} \\ \frac{m_1}{2} & \textit{if } \mathcal{C} \textit{ is nonsplit and } m_1 \textit{ is even,} \\ 0 & \textit{otherwise,} \end{cases}$$

where m_3 corresponds to the rational node (and $m_1 = m_2$).

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Proof. The proof of (a) follows easily from Proposition 9.4.

For the other cases, note that in the nonsplit case some power of Frobenius acts as negation on the component group $\Phi(\bar{\mathfrak{k}})$, so the only elements of $\Phi(\mathfrak{k})$ are elements of order 2 in $\Phi(\bar{\mathfrak{k}})$, which correspond to $[B_{m_1/2} - C_{m_2/2}]$ if m_1 and m_2 are even $\big($where μ takes the value $\frac{1}{4}(m_1 + m_2)\big)$, and similarly with the obvious cyclic permutations.

In the situation of (c), we must have $m_1 = m_2$. If $P = [(P_1) - (P_2)] \in J(k)$ and $P_1 \in C(\bar{k})$ maps to one of the conjugate nodes, then P_2 must map to the other, so all $P \in J(k)$ must map to a component of the form $[B_i - C_j]$ or $[D_i - D_j]$. Now the result in the split case follows from a case distinction depending on whether $m_1 \leq m_3$ or not. In the nonsplit case, the only element of order 2 that is defined over $\mathfrak{k}$ is $[B_{m_1/2} - C_{m_1/2}]$ if it exists.

In the situation of (d), the group $\Phi(\mathfrak{k})$ is of order 3 (generated by $[E - A]$) in the split case and trivial in the nonsplit case. $\square$

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For example: Get a proof of Mordell's Conjecture into Mathlib!

Quick Live Demo

```
import Mathlib

open Nat

theorem infinitely_many_primes : ∀ n : ℕ, ∃ p > n, p.Prime := by
  intro n
  let N := n ! + 1
  let p := N.minFac -- smallest prime divisor of `N = n! + 1`
  use p -- this will be the witness for the existential statement
  have hp : p.Prime := by -- first show that `p` is prime
    apply minFac_prime -- `N.minFac` is prime if `N ≠ 1`
    have : n ! ≠ 0 := factorial_ne_zero n
    omega -- tactic for solving linear arithmetic on `ℕ` and `ℤ`
  constructor -- split the conjunction
  · -- prove `p > n`
    by_contra! h -- assume that `p ≤ n`
    have hdvd : p ∣ n ! := (Prime.dvd_factorial hp).mpr h
    have hdvd' : p ∣ N := minFac_dvd N
    have : p ∣ 1 := (Nat.dvd_add_iff_right hdvd).mpr hdvd'
    exact hp.not_dvd_one this -- contradiction to `¬ p ∣ 1`
  · exact hp -- use proof of `p.Prime`
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So everything that follows is very preliminary
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- Once curves are there, points are easy
  ($\mathrm{Mor}_{\mathrm{Spec}\,K}(\mathrm{Spec}\,K, X)$ / places with residue field $= K$)

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But of course, we also want to have the other results from Faltings's original paper eventually!

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Mathlib currently contains more than 80 000 definitions
and more than 150 000 lemmas and theorems.

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plus diophantine approximation:

- Siegel's Lemma (over K)

The Hard Part: Vojta's Inequality

About 24 pages (Chapter 11) of [Bombieri-Gubler],
using a bunch of serious algebraic geometry, e.g.,

- Riemann-Roch on X and $X \times X$
- Intersection theory on $X \times X$
- The relation between (very ample) divisors and projective embeddings, description of global sections
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- Roth's Lemma

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Maybe better automation and/or AI methods will help speed up things

Thank You!