Algebraic points on curves

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I completed this work on the ancestral unceded lands of the **Duwamish, Suquamish, Tulalip and Muckleshoot** nations.

Algebraic points on curves

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Based in part on joint work with:

- Bourdon, Ejder, Liu, and Odomodu 2019
- Vogt `arXiv:2406.14353`
- Balçik, Chan, and Liu (*ongoing*)

The Mordell conjecture

Let C be a nice curve over a number field k.

If the genus of C is at least 2,

then $C(k)$ **is finite.**

Geometry controls arithmetic!

The Mordell conjecture (Faltings, 1983)

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What does this say about the arithmetic of C?

The Mordell conjecture (Faltings, 1983)

Let C be a nice curve over a number field k.

If the genus of C is at least 2,

then $C(k)$ is a **proper Zariski closed subset.**

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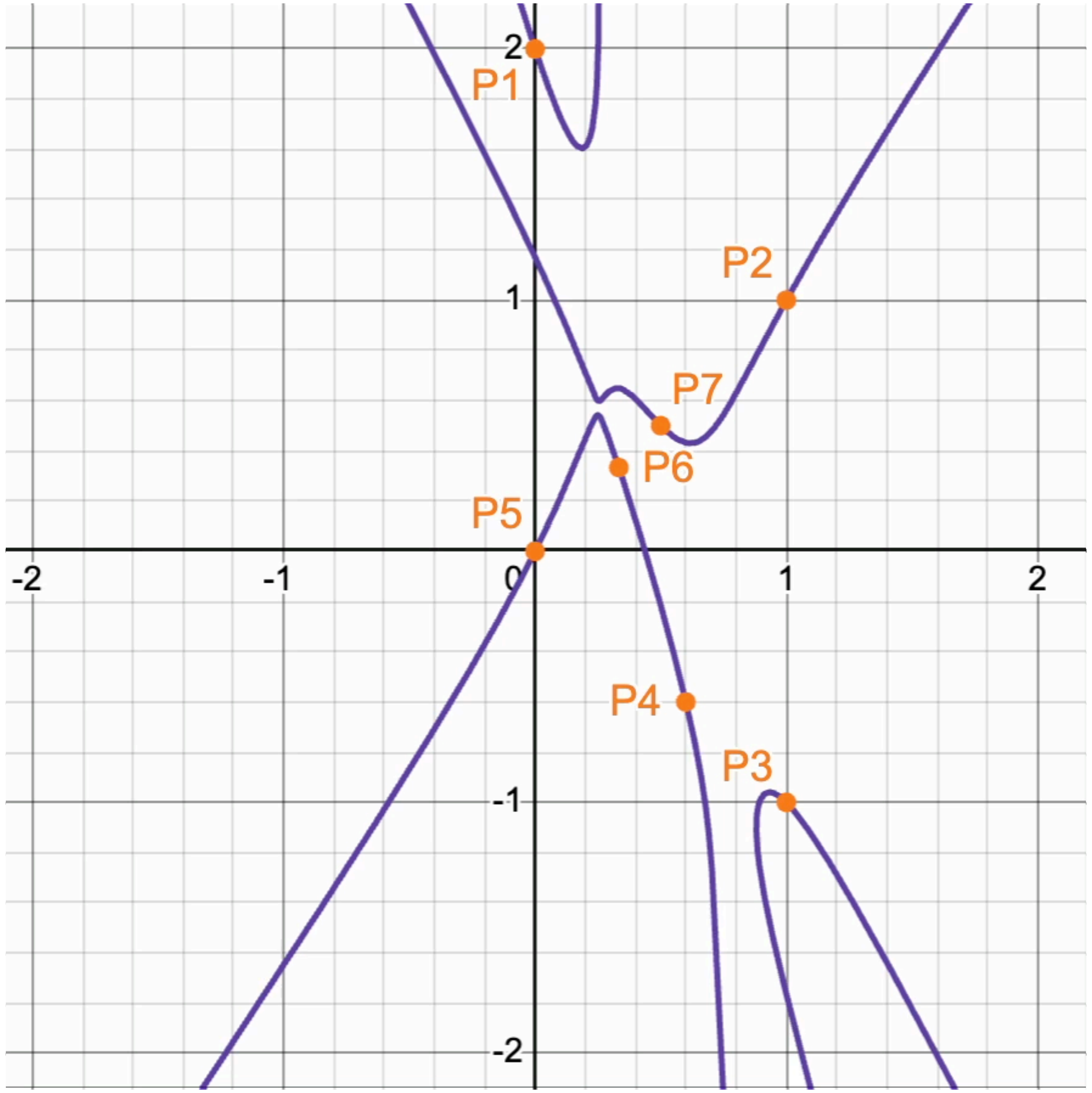
Mordell Conj. If the genus of C is at least 2,

then $C(k)$ is a **proper Zariski closed subset.**

What does this say about the arithmetic of C?

$C(k)$ reveals **very little** about C!



The Mordell conjecture: ~~*100 years later*~~

Geometry controls arithmetic, yet
$C(k)$ reveals **very little** about C**!**

Can we understand the arithmetic of **all** of C?

All closed $x \in C$, with $\mathbf{k}(x)$ $\longleftrightarrow$ $C(\bar{k}) \circlearrowleft \mathrm{Gal}(\bar{k}/k)$

closed $x \in C \longleftrightarrow$ a $\mathrm{Gal}(\bar{k}/k)$-orbit

$\mathbf{k}(x) \longleftrightarrow$ field of definition of $y \in C(\bar{k})/_{\simeq}$

The Mordell conjecture: ~~*100 years later*~~

Can we understand the arithmetic of **all** of C?

Can we understand **all** closed points of C?

… a Zariski dense set of closed points?

closed $x \in C \longleftrightarrow$ a $\mathrm{Gal}(\bar{k}/k)$-orbit

$\mathbf{k}(x) \longleftrightarrow$ field of definition of $y \in C(\bar{k})/\simeq$

The Mordell conjecture: *100 years later*

Can we understand a Zariski dense set of closed points?

Not Zariski dense

Zariski dense

closed $x \in C \longleftrightarrow$ a $\mathrm{Gal}(\bar{k}/k)$-orbit

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Visualizations inspired by Hector Pasten

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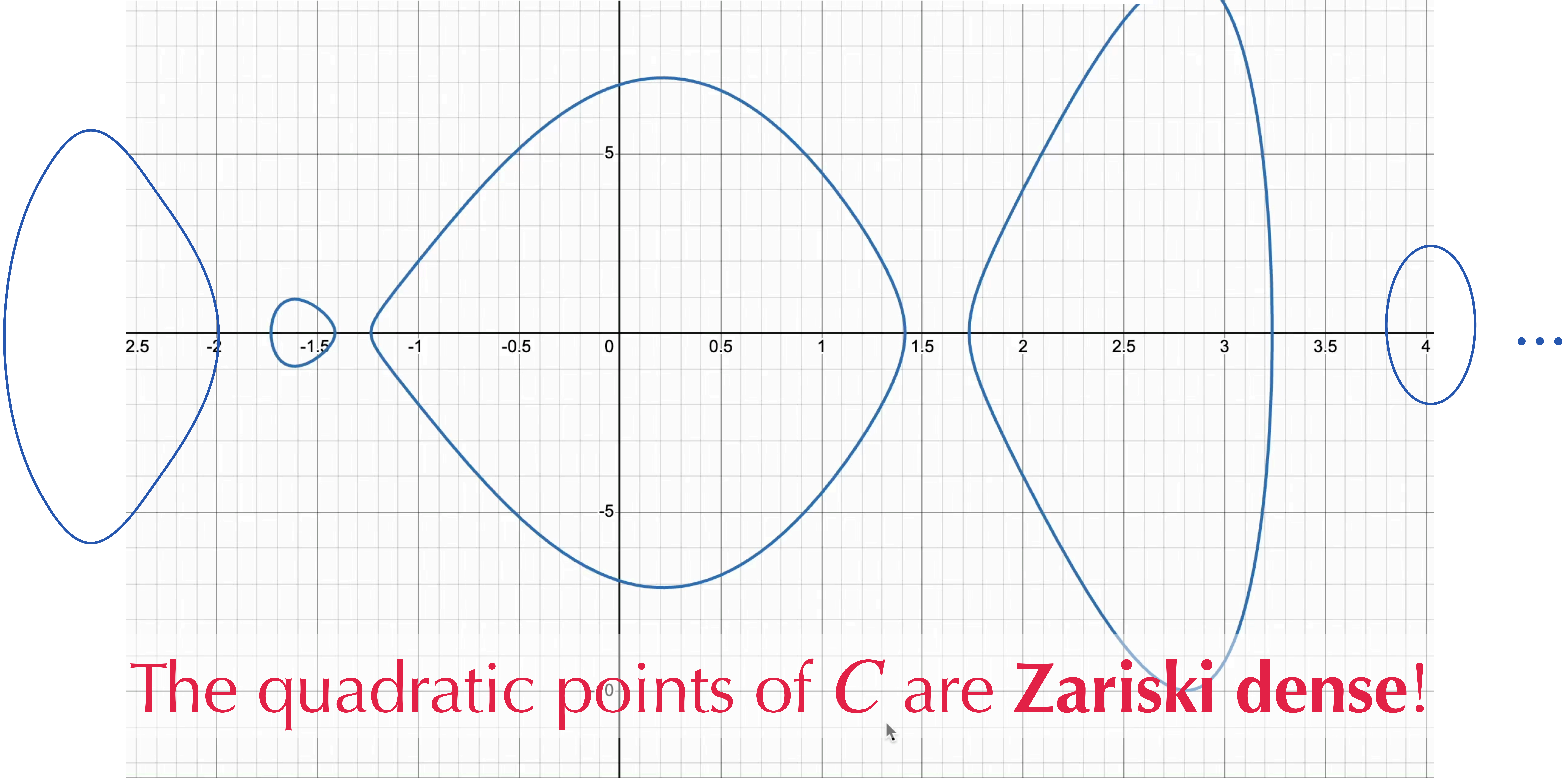
Visualizations inspired by Hector Pasten

$$C : y^2 = -2(x^2-2)(x^2-3)(x^2-2x-4)\cdots h(x)$$



The quadratic points of C are **Zariski dense**!

The Mordell conjecture: ~~100 years later~~

Can we understand a Zariski dense set of closed points?

Not Zariski dense

Zariski dense

closed $x \in C \longleftrightarrow$ a $\mathrm{Gal}(\bar{k}/k)$-orbit

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Visualizations inspired by Hector Pasten

What if $C \to \mathbb{P}^1$ has degree $d>2$?

What if $C \to \mathbb{P}^1$ has degree $d > 2$?

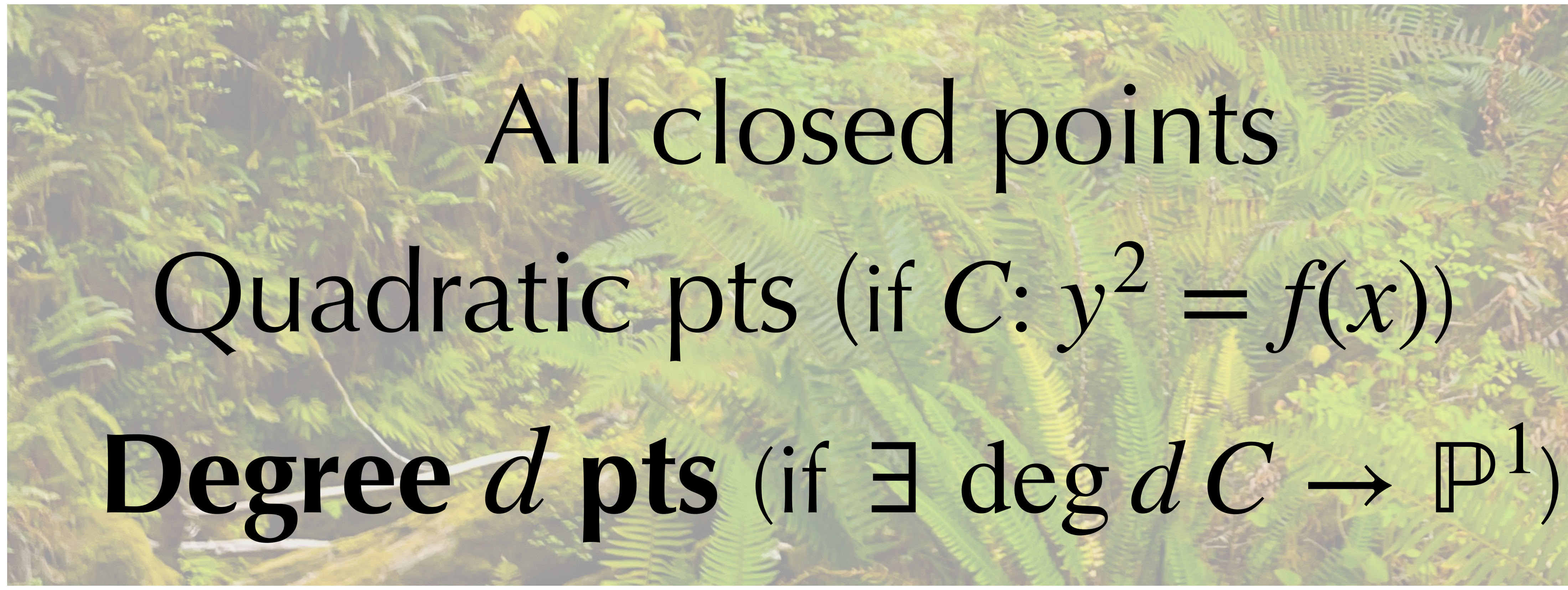
<u>Hilbert's Irreducibility Theorem</u>

The fibers over $\mathbb{P}^1(k)$ that are irreducible are **Zariski dense** on C.

The Mordell conjecture: *100 years later*

Can we understand a Zariski dense set of closed points?

Not Zariski dense



Zariski dense

Visualizations inspired by Hector Pasten

Definition The **density degree set** is

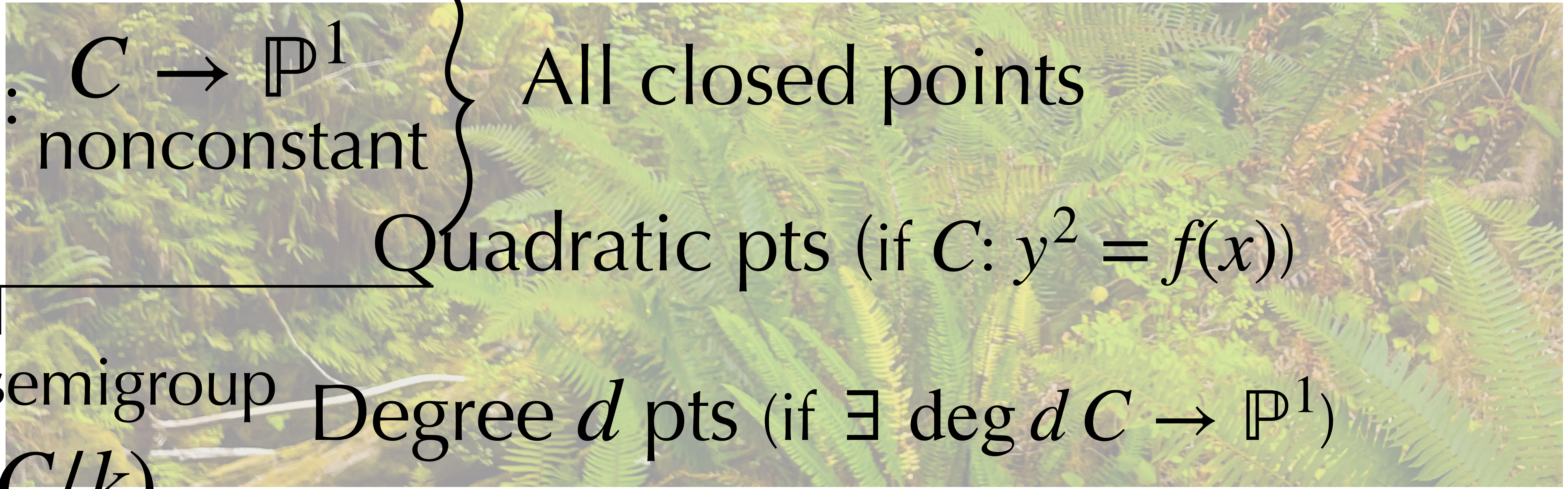
$$\delta(C/k) := \left\{ d \in \mathbb{N} : \begin{array}{l}\text{degree } d \text{ points are} \\ \text{Zariski dense on } C\end{array} \right\}$$

$\cup$ ⟵ Can this containment be **strict**?

$$\left\{ \deg(C \to \mathbb{P}^1) : \begin{array}{c} C \to \mathbb{P}^1 \\ \text{nonconstant} \end{array} \right\}$$

Lüroth semigroup

$\delta_{\mathbb{P}^1}(C/k)$



All closed points

Quadratic pts (if $C: y^2 = f(x)$)

Degree d pts (if $\exists$ deg d $C \to \mathbb{P}^1$)

Zariski dense

$$\delta(C/k) := \left\{ d \in \mathbb{N} : \begin{array}{c}\text{degree } d \text{ points are} \\ \text{Zariski dense on } C\end{array} \right\}$$

$$\left\{ \begin{array}{c}\text{degree } d \text{ points} \\ \text{on } C\end{array} \right\} \subset \left\{ \begin{array}{c}\text{degree } d \text{ 0-dim'l} \\ \text{subschemes of } C\end{array} \right\} = \mathrm{Hilb}^d_C = \mathrm{Sym}^d_C$$

If $d \in \delta(C/k)$

$$\Downarrow$$

$\exists$ positive dim'l $Z \subset \mathrm{Sym}^d_C$ with Zariski dense k-points

What are the positive dim'l $Z \subset \mathrm{Sym}^d_C$ ~~with Zariski dense k-points~~?

$$\left\{\begin{array}{c}\text{degree } d \text{ effective}\\ \text{divisors on } C\end{array}\right\} = \left\{\begin{array}{c}\text{degree } d \text{ 0-dim'l}\\ \text{subschemes of } C\end{array}\right\} = \mathrm{Hilb}^d_C = \mathrm{Sym}^d_C$$

$$\left\{\begin{array}{c}\text{degree } d \text{ effective}\\ \text{divisors on } C\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{degree } d \text{ divisor}\\ \textit{classes} \text{ on } C\end{array}\right\}$$

$$\mathrm{Sym}^d_C \xrightarrow{\ \rho\ } \mathrm{Pic}^d_C$$

$$\left\{\begin{array}{c}\text{degree } d \text{ divisor}\\ \textit{classes} \text{ on } C\end{array}\right\} \longleftrightarrow \mathrm{Pic}^d_C$$

~~What are the positive dim'l $Z \subset \mathrm{Sym}^d_C$ with Zariski dense k-points?~~

Assume $\dim \rho(Z) = 0$

$Z = |D|$ give $\mathbb{P}^1$-parameterized points

$\subset Z |D| \simeq \mathbb{P}^N \hookleftarrow \mathbb{P}^1$

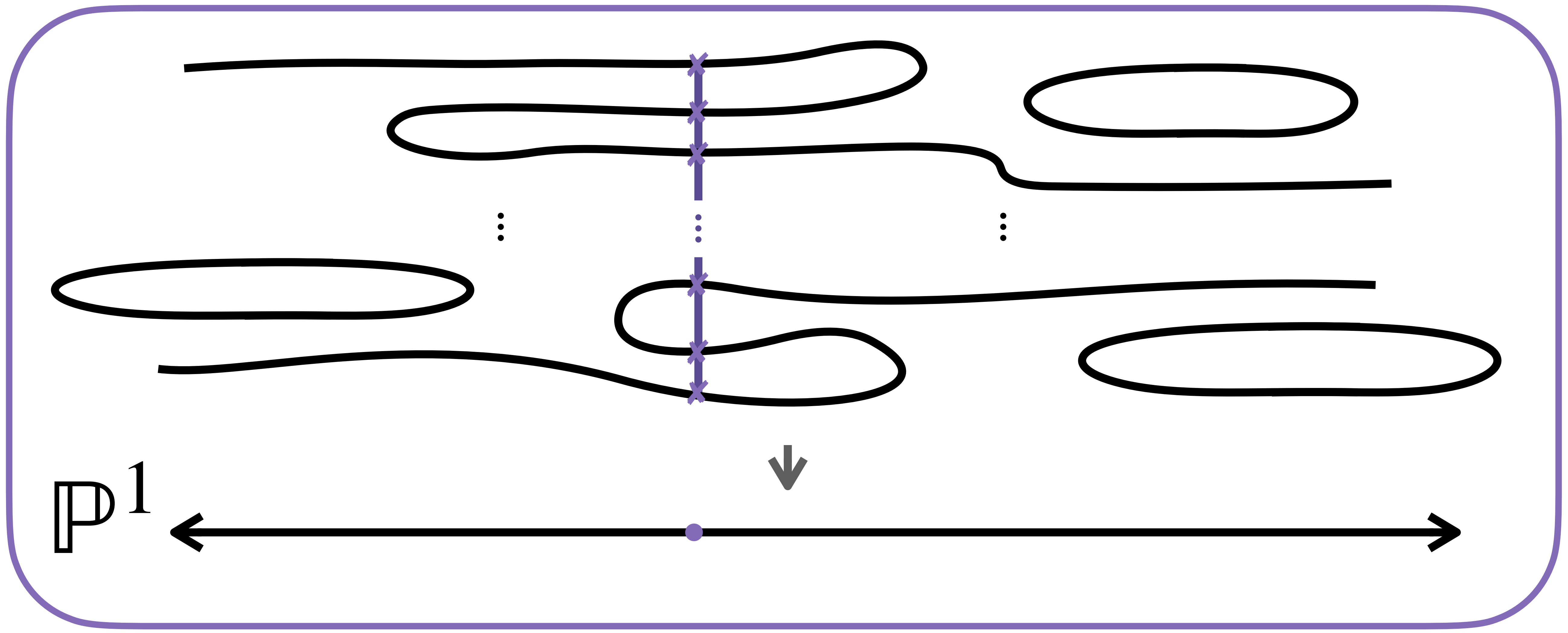
$N \geq 1$

Sym^d_C

ρ

$\bullet\ [D]$

Pic^d_C



What are the positive dim'l $Z \subset \mathrm{Sym}^d_C$ ~~with Zariski dense k-points~~?

Assume $\dim \rho(Z) = 0$

$$Z \subset |D| \simeq \mathbb{P}^N \hookleftarrow \mathbb{P}^1 \qquad \mathrm{Sym}^d_C$$

$$\downarrow \qquad\qquad \downarrow \rho$$

$$\bullet\ [D] \qquad\qquad \mathrm{Pic}^d_C$$

A closed point $x \in C$ is $\mathbb{P}^1$**-parameterized** if any (⇔all) of the following hold:

- $\exists \pi\colon C \to \mathbb{P}^1$ with $\pi(x) \in \mathbb{P}^1(k)$ & $\deg(\pi) = \deg(x)$;
- $\exists\ \mathbb{P}^1 \hookrightarrow \mathrm{Sym}^d_C$ whose image contains x;
- $\mathrm{h}^0(C, \mathcal{O}(x)) \geq 2$.

Otherwise, x is $\mathbb{P}^1$-isolated.

~~What are the positive dim'l $Z \subset \mathrm{Sym}^d_C$ with Zariski dense k-points?~~

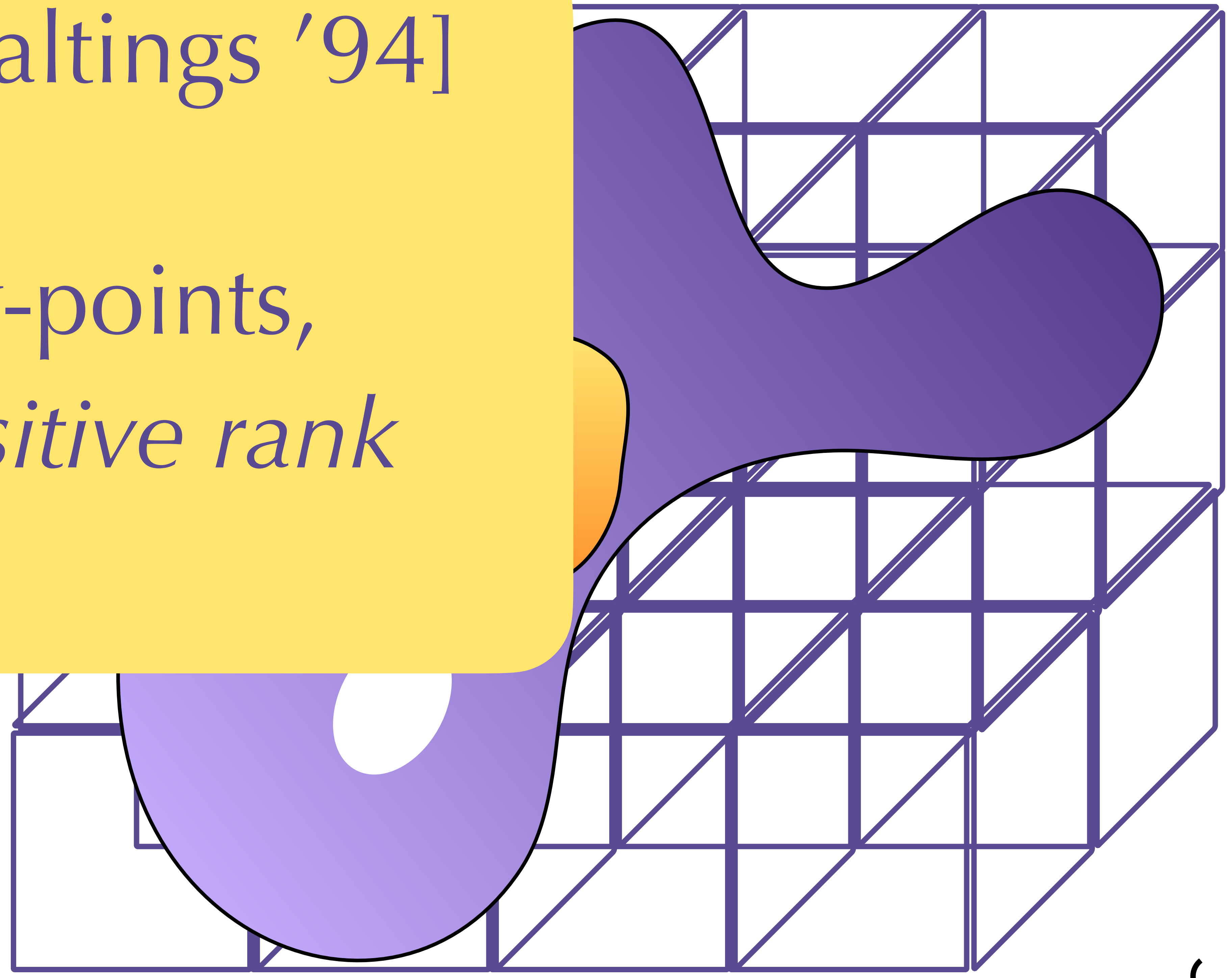
Assume $\dim \rho(Z) > 0$

Sym^d_C

$\rho \downarrow$

Mordell-Lang Conjecture [Faltings '94]

If $Y \subset A$ has Zariski dense k-points, then Y is a translate of a *positive rank abelian subvariety*.



Pic^d_C

Has Zariski dense k-points!

$\rho(Z) \hookrightarrow W^d := \rho(\mathrm{Sym}^d_C) \hookrightarrow \mathrm{Pic}^d_C$

~~What are the positive dim'l $Z \subset \mathrm{Sym}^d_C$ with Zariski dense k-points?~~

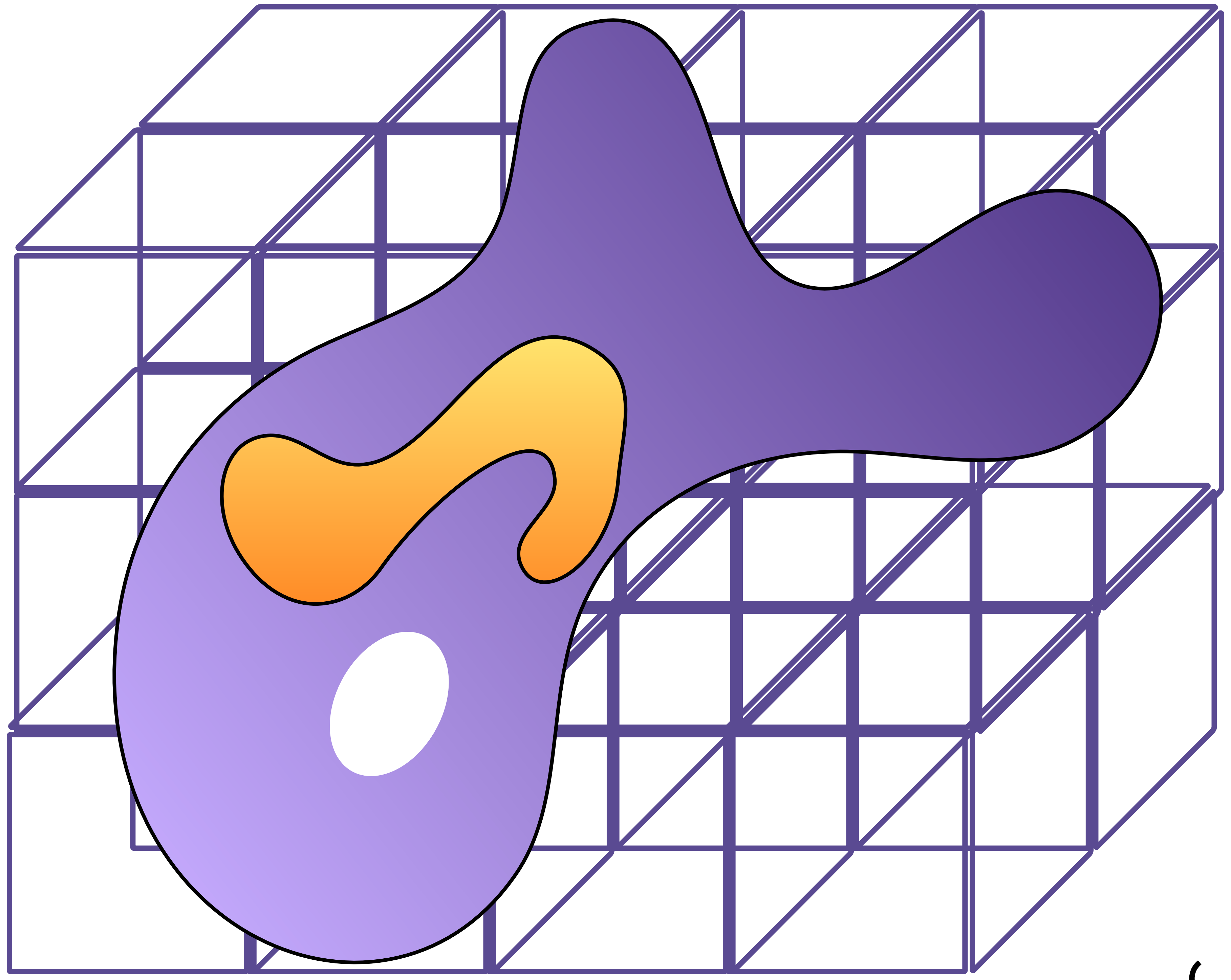
Assume $\dim \rho(Z) > 0$

Mordell-Lang Conj.
[Faltings '94]

$\Downarrow$

$\rho(Z)$ is a positive rank abelian variety

Has Zariski dense k-points!



$$\mathrm{Sym}^d_C \xrightarrow{\ \rho\ } \mathrm{Pic}^d_C$$

$$\rho(Z) \hookrightarrow W^d := \rho(\mathrm{Sym}^d_C) \hookrightarrow \mathrm{Pic}^d_C$$

~~What are the positive dim'l $Z \subset \mathrm{Sym}^d_C$ with Zariski dense k-points?~~

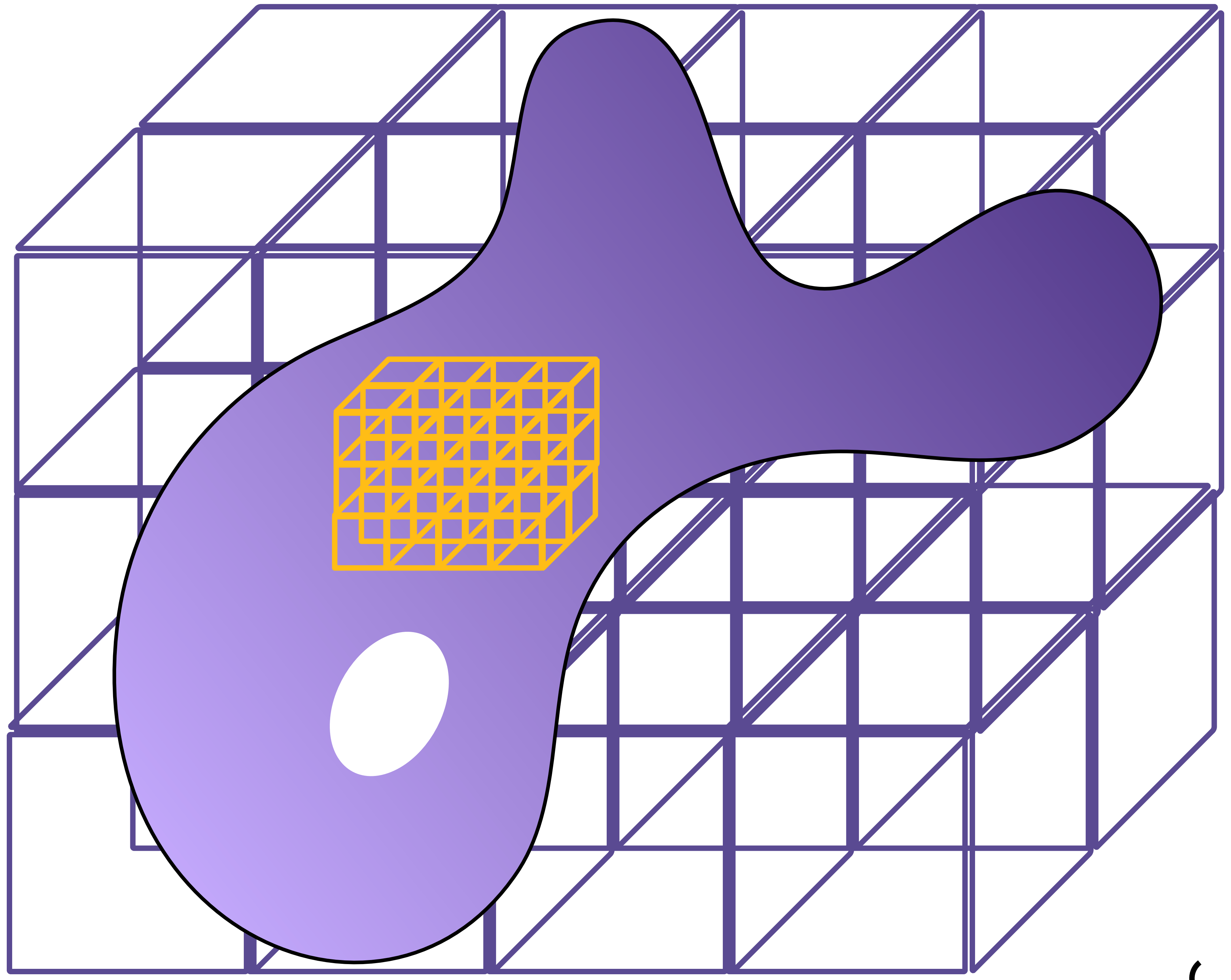
Assume $\dim \rho(Z) > 0$ **Such Z give AV-parameterized points**

Mordell-Lang Conj.
[Faltings '94]

$\Downarrow$

$\rho(Z)$ is a positive rank abelian variety

Has Zariski dense k-points!



$$\mathrm{Sym}^d_C \xrightarrow{\rho} \mathrm{Pic}^d_C$$

$$\rho(Z) \hookrightarrow W^d := \rho(\mathrm{Sym}^d_C) \hookrightarrow \mathrm{Pic}^d_C$$

What are the positive dim'l $Z \subset \mathrm{Sym}^d_C$ ~~with Zariski dense k-points~~ ?

Assume $\dim \rho(Z) > 0$

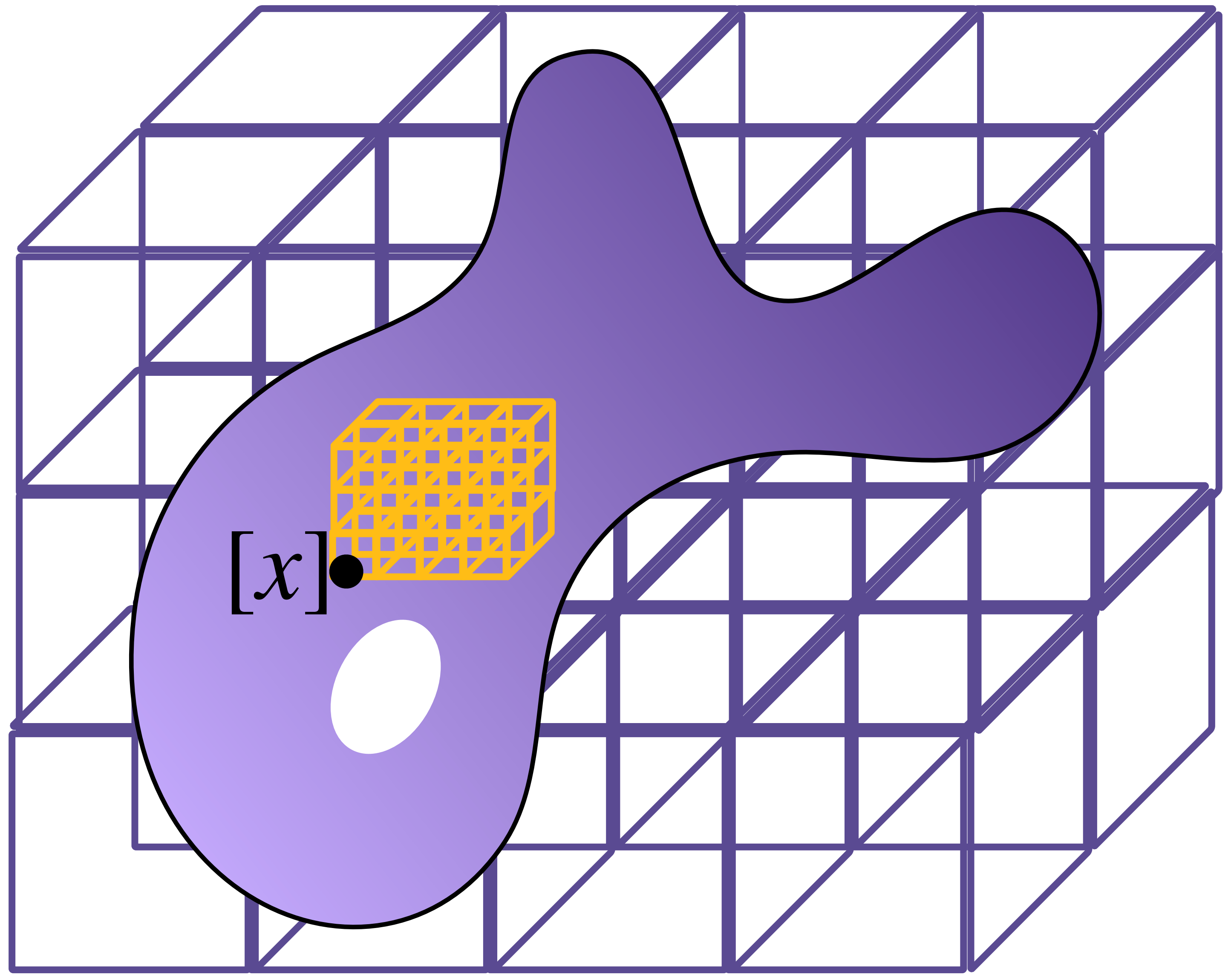
A closed point $x \in C$ is

AV-parameterized

if $\exists$ a positive rank abelian subvariety $B \subset \mathrm{Pic}^0_C$ such that $[x] + B \subset W^d$.

Otherwise, x is AV-isolated.

$$\mathrm{Sym}^d_C \xrightarrow{\ \rho\ } \mathrm{Pic}^d_C$$



$W^d := \rho(\mathrm{Sym}^d_C)$

Examples of curves with degree d AV-parameterized points

1. $C \to E$ degree d morphism, E positive rank elliptic curve, and $\exists P \in E(k)$ such that C_P irreducible.

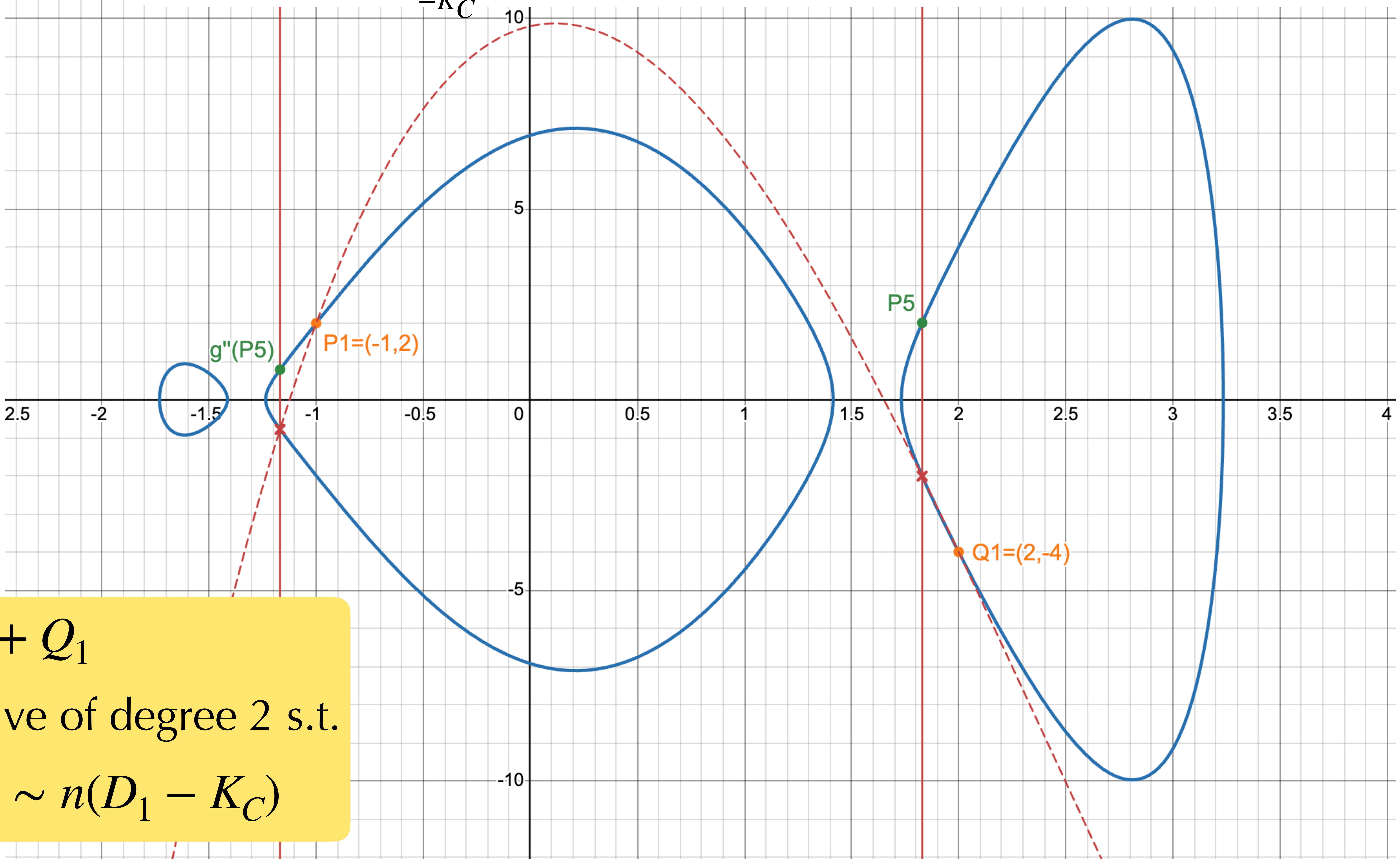
2. $C \subset \mathrm{Sym}^2_{E'}$, E positive rank elliptic curve, $C \sim (d+m)H - mF$ for some $d, m \in \mathbb{N}$, $1 \leq m \leq d$ and $C \cap H_x$ is irreducible for some H_x representing H. [Debarre-Fahlaoui '93; Kadets-Vogt]

3. $d = 2$, C genus 2 such that Jac_C is simple and has positive rank.

$$W^2_C = \mathrm{Pic}^2_C \xrightarrow[T_{-K_C}]{\sim} \mathrm{Pic}^0_C \text{ and } \mathrm{Sym}^2_C \to \mathrm{Pic}^2_C \text{ one-to-one away from } K_C$$

3. $d = 2$, C genus 2 such that Jac_C is simple and has positive rank.

$W_C^2 = \mathrm{Pic}_C^2 \xrightarrow[T_{-K_C}]{\sim} \mathrm{Pic}_C^0$ and $\mathrm{Sym}_C^2 \to \mathrm{Pic}_C^2$ one-to-one away from K_C



$D_1 = P_1 + Q_1$

D_n effective of degree 2 s.t.

$D_n - K_C \sim n(D_1 - K_C)$

$$\delta(C/k) := \left\{ d \in \mathbb{N} : \begin{array}{l} \text{degree } d \text{ points are} \\ \text{Zariski dense on } C \end{array} \right\}$$

If $d \in \delta(C/k)$, then $\exists$ positive dim'l $Z \subset \mathrm{Sym}^d_C$ with Zariski dense k-points

A closed point $x \in C$ is $\mathbb{P}^1$-**parameterized** if…

Otherwise, x is $\mathbb{P}^1$-isolated.

A closed point $x \in C$ is **AV-parameterized** if …

Otherwise, x is AV-isolated.

parametrized = $\mathbb{P}^1$- OR AV-parametrized

isolated = $\mathbb{P}^1$- AND AV-isolated

Theorem [Bourdon, Ejder, Liu, Odumodu, Viray 2019]

Corollary of Faltings' 1994 proof of Mordell-Lang

Let C/k be a nice curve.

1. $d \in \delta(C/k)$, i.e., the degree d points are Zariski dense on C

$$\Updownarrow$$

$\exists$ degree d parameterized point.

parametrized = $\mathbb{P}^1$- OR AV-parametrized

isolated = $\mathbb{P}^1$- AND AV-isolated

Theorem [Bourdon, Ejder, Liu, Odumodu, Viray 2019]

Corollary of Faltings' 1994 proof of Mordell-Lang

Let C/k be a nice curve.

1. $\delta(C/k) = \delta_{\mathbb{P}^1}(C/k) \cup \delta_{\mathrm{AV}}(C/k)$, where

$$\delta_{\mathbb{P}^1}(C/k) := \left\{\deg(x) : x \in C,\ \mathbb{P}^1\text{-parameterized}\right\}$$

$$\delta_{\mathrm{AV}}(C/k) := \left\{\deg(x) : x \in C,\ \text{AV-parameterized}\right\}$$

parametrized = $\mathbb{P}^1$- OR AV-parametrized

isolated = $\mathbb{P}^1$- AND AV-isolated

Properties of $\delta(C/k)$ (see [Viray, Vogt])

1. $\delta(C/k)$ contains all sufficiently large multiples of $\mathrm{ind}(C/k)$.
2. If $d \in \delta_{\mathrm{AV}}(C/k)$ and $n \geq 2$, then $nd \in \delta_{\mathrm{AV}}(C/k) \cap \delta_{\mathbb{P}^1}(C/k)$.

   In particular, $\delta(C/k)$ is closed under multiplication by $\mathbb{N}$.

Corollaries

- $\mathrm{gon}(C/k) \leq 2 \min \delta(C/k)$. ([Abramovich, Harris] and [Frey])
- If k'/k is a finite extension, then $\delta(C/k) \subset \delta(C/k')$.

$\delta_{\mathbb{P}^1}(C/k) := \{\deg(x) : x \in C, \mathbb{P}^1\text{-parameterized}\}$

$\delta_{\mathrm{AV}}(C/k) := \{\deg(x) : x \in C, \text{AV-parameterized}\}$

Theorem [Bourdon, Ejder, Liu, Odumodu, Viray 2019]

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2. There are finitely many isolated points on C. Reveal little about C!

parametrized = $\mathbb{P}^1$- OR AV-parametrized

isolated = $\mathbb{P}^1$- AND AV-isolated

Theorem [Bourdon, Ejder, Liu, Odumodu, Viray 2019]

Corollary of Faltings' 1994 proof of Mordell-Lang

Let C/k be a nice curve.

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$$\delta_{\mathrm{AV}}(C/k) := \left\{\deg(x) : x \in C,\ \text{AV-parameterized}\right\}$$

2. $\exists$ open $U \subset C$ s.t. every closed $x \in U$ is parameterized.

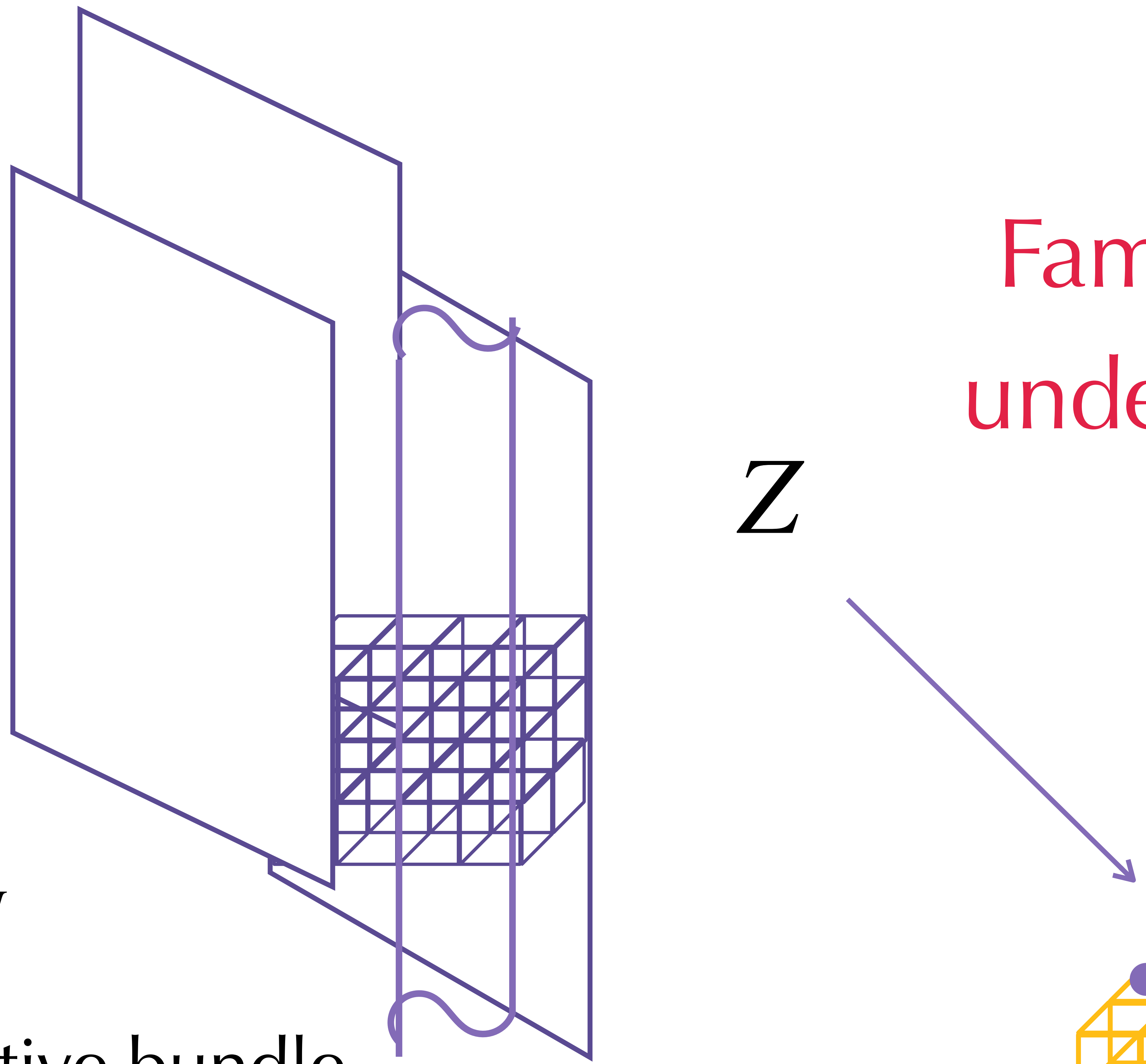
param ... ated

Can we understand **all** parameterized points on C?

as a scheme, i.e., with $\mathbf{k}(x)$

Can we understand **all** parameterized points on C?

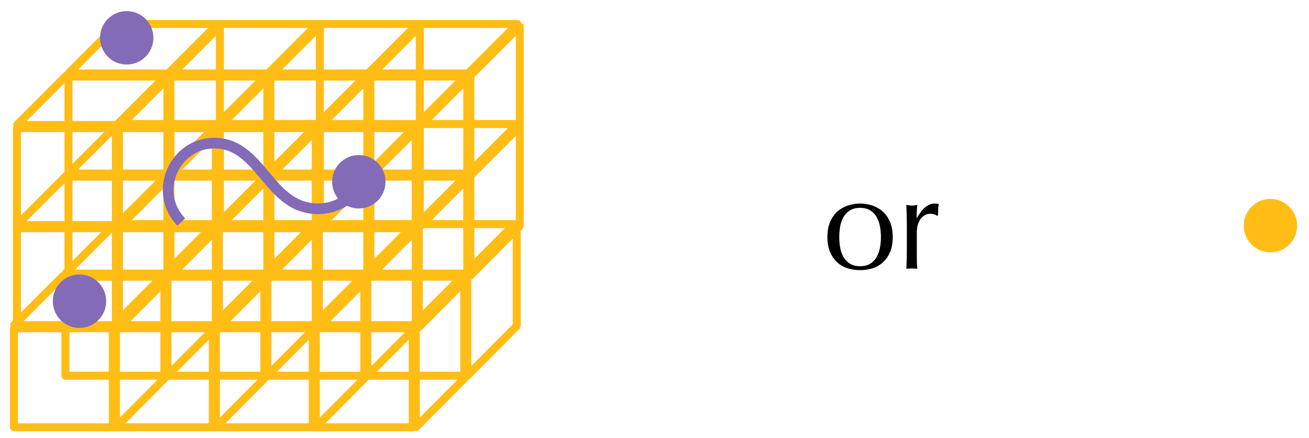
For a fixed degree, parametrized points arise in finitely many families.



Families have a well understood geometric structure!

Fibers with k-points are $\simeq \mathbb{P}^N$

As $d \to \infty$, $Z \to A$ is a projective bundle



Can we understand **all** parameterized points on C?

For a fixed degree, parametrized points arise in finitely many families.

Given a **parameterized** point $x \in C$,

how does $\mathbf{k}(x)$ vary in the parameterization?

$\mathbb{P}^1$-

Given a **parameterized** point $x \in C$,

how does $\mathbf{k}(x)$ vary in the parameterization?

Meta Theorem [Balçik, Chan, Liu, Viray; *in progress*]

Let $C \xrightarrow{\pi} \mathbb{P}^1$ be a degree d map.

If $v \in \Omega_k$ with $\#\mathbb{F}_v >> 0$ and let L/k_v be an extension

compatible with the geometry of π,

Then there exists a $t \in \mathbb{P}^1(k)$ such that $\mathbf{k}(C_t) \otimes_k k_v$ contains a maximal subfield isomorphic to L.

Theorem [Balçik, Chan, Liu, Viray; *in progress*]

Let $C \xrightarrow{\pi} \mathbb{P}^1$ be cyclic of degree d & s.t. all ramification points have ram. index d.

Then

$\forall\, v \in \Omega_k$ with $\#\mathbb{F}_v >> 0$, $\forall\, f \,|\, d$, and $\forall$ totally ramified degree d ext′ns L/k_v:

- $\exists t \in \mathbb{P}^1(k)$ such that $\mathbf{k}(C_t)$ has $\dfrac{d}{f}$ places above v, each with inertia degree f.
- $\exists\, t \in \mathbb{P}^1(k)$ such that $\mathbf{k}(C_t) \otimes_k k_v \simeq L \iff \mathbb{P}^1(\mathbb{F}_v)$ contains a branch point.

Theorem [Balçik, Chan, Liu, Viray; *in progress*]

Let $C \xrightarrow{\pi} \mathbb{P}^1$ be a degree d map whose Galois closure is an S_d extension & s.t. all branch points have a unique ramification about with ram. index 2. Then $\forall\, v \in \Omega_k$ with $\#\mathbb{F}_v >> 0$, $\forall\, (f_i) \vdash d$:

- $\exists t \in \mathbb{P}^1(k)$ such that $\mathbf{k}(C_t)$ is unramified at v with inertia degrees (f_i).
- $\exists\, t \in \mathbb{P}^1(k)$ such that $\mathbf{k}(C_t)$ is ramified at $v \Leftrightarrow\ \mathbb{P}^1(\mathbb{F}_v)$ contains a branch point.
- Furthermore, for any $t \in \mathbb{P}^1(k)$, there is at most one $w \mid v$ that is ramified and it has $e(w/v) = 2$.

Further Directions

1. Classification of curves with a fixed minimum density degree. (see [Harris—Silverman, Abramovich—Harris, Kadets—Vogt])
2. Geometric restrictions from low degree parameterized points.
3. Galois-theoretic properties in $\mathbb{P}^1$- and AV-parametrizations. (See [Khawaja—Siksek])
4. Uniform bounds for the number of isolated points.
5. Variation of residue fields in AV-parametrizations.