Mordell's conjecture in positive characteristic and descent obstructions

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The Mordell conjecture 100 years later, MIT

July, 2024

Abstract

I will give a brief historical overview of the Mordell conjecture over function fields of positive characteristic (mainly). Then, I'll discuss its relationship with descent obstructions for rational points and present a recent result obtained with B. Creutz, which completes my earlier work with B. Poonen, showing that finite descent obstructions fully characterize the rational points on non-isotrivial curves over function fields.

Preliminaries

k field. K/k function field (finitely generated extension of transcendence degree 1).

C a (smooth, irreducible, projective) curve defined over K.

C is **isotrivial** if there exists L/K finite and a curve C_0/k such that C and C_0 are isomorphic over L.

Conjecture

(Mordell conjecture for function fields) If C is non-isotrivial of genus $g \geq 2$, then $C(K)$ is finite.

(Proved by Manin, Grauert and Samuel)

Lang

In this connection, the Mordell conjecture becomes a conjecture in algebraic geometry, and it is worth while to make further comments on it here. Let k be as above, $K = k(t)$ a function field over k, where t is the generic point of a variety T, and let C be a curve of genus $\geqq 2$, defined over K. Then $C = C_t$ can be viewed as the generic member of an algebraic family. *The conjecture then asserts that if* C_t *has infinitely many rational points in* $k(t)$ *(cross sections of the parameter variety* T *in the graph of the family), then* C_t *is birationally equivalent over* $k(t)$ *to a curve* C_0 *defined over* k*, and all but a finite number of these points arise from points of* C_0 *in* k.

S. Lang, Integral points on curves, P. M. IHES, 1960. (Char. 0)

Some 20th century events

- **1960s**: Manin and Grauert (independently) prove the Mordell conjecture for function fields of characteristic zero
- **1960s**: Samuel extends Grauert's approach to positive characteristic.
- **1970s**: Szpiro proves Shafarevich's conjecture and an effective version of Mordell's conjecture in positive characteristic.
- **1980s**: Faltings proves Mordell over number fields. Storrs conference August 1984.
- **1990s**: V. (for ordinary curves) and Abramovich-V. (in general) give a proof of Mordell in positive characteristic that extends to Mordell-Lang.
- **1990s**: Hrushovski proves Mordell-Lang for subvarieties of abelian varieties over function fields.

Idea of our approach

Embed C in its Jacobian J. The Frobenius $F : J \to J^{(p)}$ has a dual isogeny $V : J^{(p)} \to J$. Let $C_1 := V^{-1}(C)$. We have:

$$V(J^{(p)}(K^p)) = V(F(J(K))) = pJ(K)$$

is of finite index in $J(K)$. If $C(K)$ is infinite, then (after a translation) C_1 has infinitely many K^p points, hence descends to K^p and so does C (needs an argument, which is harder when V is not separable). Iterating leads to C isotrivial (since $\bigcap K^{p^n} = k$).

Descent obstructions

$\mathbb{A}_K$ denotes the adèles of K, $X(\mathbb{A}_K) = \prod_v X(K_v)$.

Definition 1

An N-covering of C is a torsor $D \to C$ (over K) under $J[N]$ such that the base change of $D \to C$ to K^{sep} is isomorphic to the pull back of multiplication by N on J. An adelic point on C is said to survive N-descent if it lifts to an adelic point on some N-covering of C.

(Rational points always survive N-descent)

Definition 2

An adelic point $(P_v)_v \in X(\mathbb{A}_K)$ is called Zariski dense if for any proper closed subvariety $Y \subsetneq X$, there exists v such that $P_v \notin Y$.

Descent obstructions II

Theorem

(Creutz-V.) If there is a Zariski dense adelic point on C which survives p^n-descent for all $n \geq 1$, then C is isotrivial.

(With crucial input from a result of Rössler)

Corollary

If C is not isotrivial, then $C(\mathbb{A}_K)^{\mathrm{Br}} = C(K)$. Also, $C(\mathbb{A}_K) \cap \overline{J(K)} = C(K)$, where $\overline{J(K)}$ denotes the topological closure of $J(K)$ in $J(\mathbb{A}_K)$.

The corollary was proved by Poonen and V. (2010) under additional hypotheses.

Descent obstructions III

The isotrivial case is more complicated!

We have a set $C(\mathbb{A}_{K,k}) \subset C(\mathbb{A}_K)$ which, in the constant case, is $\prod_v C(k_v)$ ($k_v =$ residue field), that is hard to pin down. Apart from this set, the rational points can be picked out from the adelic points by descent obstructions.

Theorem
(Creutz, Pajwani, V.) If C/K is isotrivial,

$$C(\mathbb{A}_K)^{\mathrm{Br}} = C(\mathbb{A}_K) \cap \overline{J(K)} = C(K) \cup (C(\mathbb{A}_{K,k}) \cap \overline{J(K)}).$$

THANK YOU